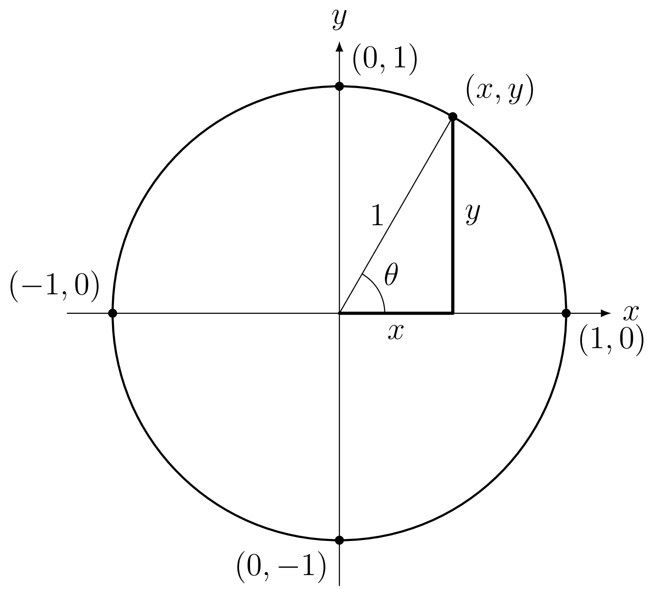
Review of Basic Trigonometry

1 Definitions of $\cos\theta$ and $\sin\theta$

Draw a circle of radius $r = 1$ centered at the origin. This is the *unit circle*. Now pick a point somewhere on the perimeter of the circle and call the coordinates of this point (x, y). A line drawn from this point to the center of the circle will be 1 unit long (since the radius of the circle is 1) and will make an angle θ (the Greek letter *Theta*) between itself an the positive x-axis; θ is positive if measured in the counter-clockwise direction and negative if measured clockwise.



Notice that for any point on the circle's perimeter, x and y will always have values between -1 and 1, and that both these values will change as θ changes. It turns out to be terrifically helpful to have functions that tell us the values of x and y given a value for θ.

Definition: Let x and y be given as described above. Then

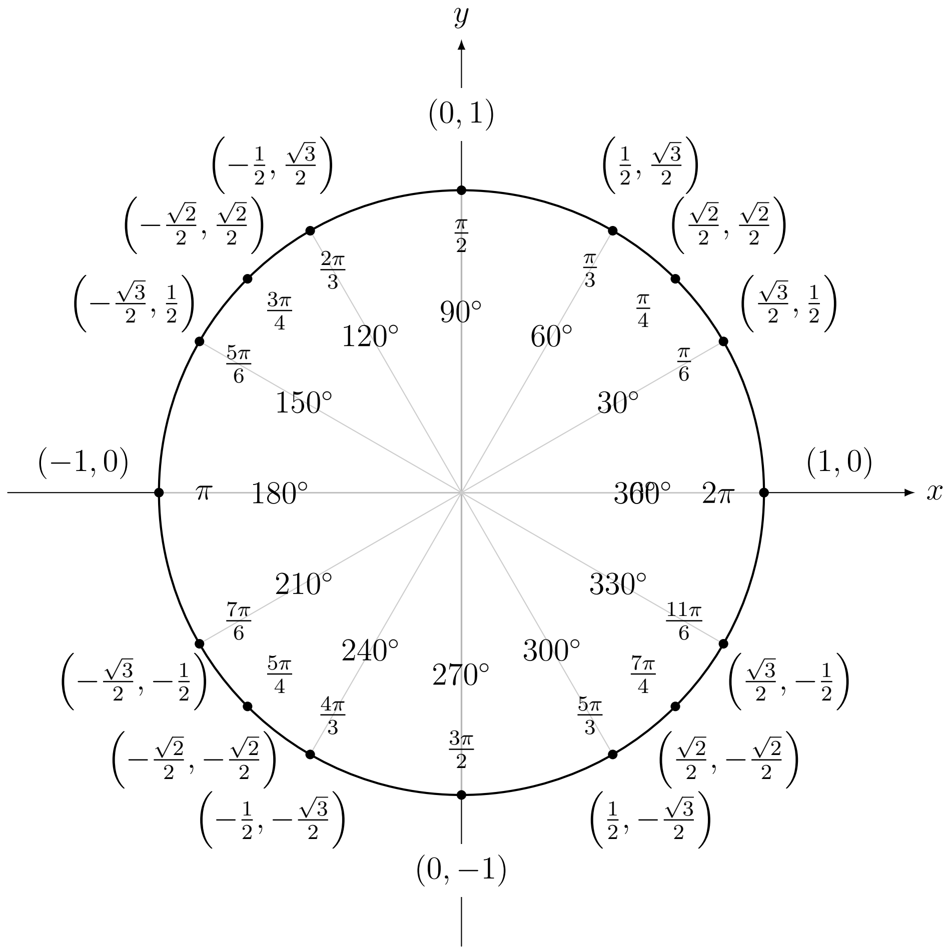
$$\cos\theta = x \qquad \text{and} \qquad \sin\theta = y$$

give the definitions of the *cosine* and *sine* functions, respectively.

2 A Unit Circle Reference

The following reference unit circle[1] can help you with the values of *cosine* and *sine* for a range of common angles. **Note:** For reasons that will become clear during the semester, we always prefer to measure angles using *radians* rather than *degrees*. Please make sure your calculator is in radian mode!

[1]Based on one from `http://www.texample.net/media/tikz/examples/TEX/unit-circle.tex`



The coordinates shown around the perimeter of the circle give the values of $\cos\theta$ and $\sin\theta$, respectively. For example, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$.

3 Four More Trigonometric Functions

There are four more trigonometric functions that are defined in terms of sine and cosine.

Definition: We define *tangent*, *cotangent*, *secant*, and *cosecant* as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}$$

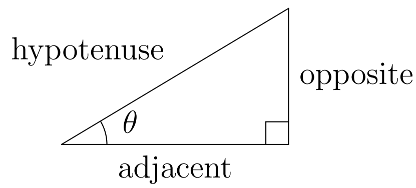
Here are a few questions to think about. You may find it helpful to refer to the unit circle diagrams on the previous pages.

1. The range of $\sin\theta$ and $\cos\theta$ is $[-1,1]$ (all values from -1 to 1). What is the range of each of these new functions?

2. Both $\sin\theta$ and $\cos\theta$ are defined for all real numbers. Are there any values of θ at which these new functions are not defined?

4 SOHCAHTOA

Let's draw a reference triangle – this is a *right* triangle with one of the other angles labeled. The side opposite the right angle is called the *hypotenuse*, the side connecting the right angle to the labeled angle is the *adjacent* side, and the remaining side is the *opposite* side (it is opposite the named angle).



The mnemonic "SOHCAHTOA" stands for "Sine – Opposite over Hypotenuse; Cosine – Adjacent over Hypotenuse; Tangent – Opposite over Adjacent." Many find this helps them remember the relationship between each of the trigonometric functions and ratios of side lengths on a reference triangle.

The **S**ine of θ is the ratio of the length of the **O**pposite side to the length of the **H**ypotenuse.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}.$$

The **C**osine of θ is the ratio of the length of the **A**djacent side to the length of the **H**ypotenuse.

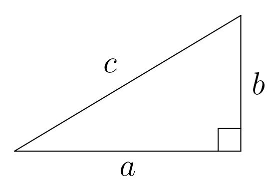
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

The **T**angent of θ is the ratio of the length of the **O**pposite side to the length of the **A**djacent side.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

5 Trigonometric Identities

Remember the Pythagorean Theorem? It says that if a, b, and c are the lengths of the sides of any right triangle with c being the length of the hypotenuse, the relationship $a^2 + b^2 = c^2$ holds.



Look back at the triangle on the first page. Using the Pythagorean Theorem we see that $y^2 + x^2 = 1^2$. However, since we defined $\cos\theta = x$ and $\sin\theta = y$, we see that

$$\sin^2\theta + \cos^2\theta = 1.$$

This is a *trigonometric identity*; a relation that is always true regardless of the value of θ. Dividing this by either $\cos\theta$ or $\sin\theta$ we have two more identities:

$$\tan^2\theta + 1 = \sec^2\theta, \qquad 1 + \cot^2\theta = \csc^2\theta.$$

Here are several more useful identities.

- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$
- $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\cos(-\theta) = \cos\theta$ (even function)
- $\sin(-\theta) = -\sin\theta$ (odd function)

The last two statements demonstrate that $\sin\theta$ and $\cos\theta$ are *odd* and *even* functions, respectively. Can you see, by looking at the unit circle, why these must be true?

5.1 Trigonometric Identities you should remember

In this and the following sections we will see how, starting with just three of the identities listed on the previous page, we can derive nearly all the commonly used trigonometric identities. The "big three" identities are

$$\sin^2\theta + \cos^2\theta = 1 \tag{1}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{2}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3}$$

Using these we can derive many other identities. Even if we commit the other useful identities to memory, these three will help be sure that our signs are correct, etc.

5.2 Two more easy identities

From equation (1) we can generate two more identities. First, divide each term in (1) by $\cos^2\theta$ (assuming it is not zero) to obtain

$$\tan^2\theta + 1 = \sec^2\theta. \tag{4}$$

When we divide by $\sin^2\theta$ (again assuming it is not zero) we get

$$1 + \cot^2\theta = \csc^2\theta. \tag{5}$$

5.3 Identities involving the difference of two angles

From equations (2) and (3) we can get several useful identities. First, recall that

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta. \tag{6}$$

From (2) we see that

$$\begin{aligned} \sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) \\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \end{aligned}$$

which, using the relationships in (6), reduces to

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{7}$$

In a similar way, we can use equation (3) to find

$$\begin{aligned} \cos(\alpha-\beta) &= \cos(\alpha+(-\beta)) \\ &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \end{aligned}$$

which simplifies to

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \tag{8}$$

Notice that by remembering the identities (2) and (3) you can easily work out the signs in these last two identities.

5.4 Identities involving products of Sines and Cosines

If we now add equation (2) to equation (7)

$$\begin{aligned} \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ +(\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta) \end{aligned}$$

we find

$$\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2\sin\alpha\cos\beta$$

and dividing both sides by 2 we obtain the identity

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha - \beta) + \frac{1}{2}\sin(\alpha + \beta). \tag{9}$$

In the same way we can add equations (3) and (8)

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ +(\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta) \end{aligned}$$

to get

$$\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2\cos\alpha\cos\beta$$

which can be rearranged to yield the identity

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta). \tag{10}$$

Suppose we wanted an identity involving $\sin\alpha\sin\beta$. We can find one by slightly modifying the last thing we did. Rather than adding equations (3) and (8), all we need to do is subtract equation (3) from equation (8):

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ -(\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta) \end{aligned}$$

This gives

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$$

or, in the form we prefer,

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta). \tag{11}$$

5.5 Double angle identities

Now a couple of easy ones. If we let $\alpha = \beta$ in equations (2) and (3) we get the two identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \tag{12}$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha. \tag{13}$$

5.6 Identities for Sine squared and Cosine squared

If we have $\alpha = \beta$ in equation (10) then we find

$$\begin{aligned} \cos\alpha\cos\alpha &= \frac{1}{2}\cos(\alpha-\alpha) + \frac{1}{2}\cos(\alpha+\alpha) \\ \cos^2\alpha &= \frac{1}{2}\cos 0 + \frac{1}{2}\cos 2\alpha. \end{aligned}$$

Simplifying this and doing the same with equation (11) we find the two identities

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha), \tag{14}$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha). \tag{15}$$

5.7 Identities involving tangent

Finally, from equations (2) and (3) we can obtain an identity for $\tan(\alpha+\beta)$:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Now divide numerator and denominator by $\cos\alpha\cos\beta$ to obtain the identity we wanted:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}. \tag{16}$$

We can get the identity for $\tan(\alpha-\beta)$ by replacing β in (16) by $-\beta$ and noting that tangent is an odd function:

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \tag{17}$$

5.8 Summary

There are many other identities that can be generated this way. In fact, the derivations above are not unique — many trigonometric identities can be obtained many different ways. The idea here is to be very familiar with a small number of identities so that you are comfortable manipulating and combining them to obtain whatever identity you need to.

6 A Short Table of Identities

1. $\sin^2\theta + \cos^2\theta = 1$

2. $\tan^2\theta + 1 = \sec^2\theta$

3. $1 + \cot^2\theta = \csc^2\theta$

4. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

5. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

6. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

7. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

8. $\sin\alpha\cos\beta = \dfrac{1}{2}\sin(\alpha-\beta) + \dfrac{1}{2}\sin(\alpha+\beta)$

9. $\cos\alpha\cos\beta = \dfrac{1}{2}\cos(\alpha-\beta) + \dfrac{1}{2}\cos(\alpha+\beta)$

10. $\sin\alpha\sin\beta = \dfrac{1}{2}\cos(\alpha-\beta) - \dfrac{1}{2}\cos(\alpha+\beta)$

11. $\sin 2\alpha = 2\sin\alpha\cos\alpha$

12. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

13. $\cos^2\alpha = \dfrac{1}{2}(1+\cos 2\alpha)$

14. $\sin^2\alpha = \dfrac{1}{2}(1-\cos 2\alpha)$

15. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

16. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$