Area Between Curves

Suppose a reservoir initially has 12.8 million gallons of water and the water flows into it at a rate of $37e^{-0.015t}$ million gallons per year (t measured in years) and water is withdrawn at a rate of $24.4e^{0.03t}$ million gallons per year.

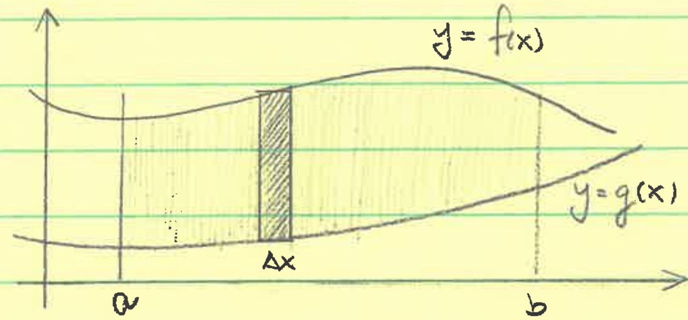
Notice that initially more water is entering the reservoir than is leaving but that this eventually changes.

1) Can we find an expression (i.e. a formula) for the amount of water in the reservoir in year T?

2) In what year will the amount of water in the reservoir be at a maximum?

To answer these questions we need to know how to compute the area between two curves...

Area Between Curves



Suppose we want to find the area of the region bounded by $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$.

All we need to do is compute the Riemann Sum of the rectangles between the two curves.

$$A \approx \sum_{i=1}^{n} \left[f(c_i) - g(c_i) \right] \Delta x$$

Where c_i is some point on $[x_{i-1}, x_i]$ and $\Delta x = \frac{b-a}{n}$ where n is the number of rectangles.

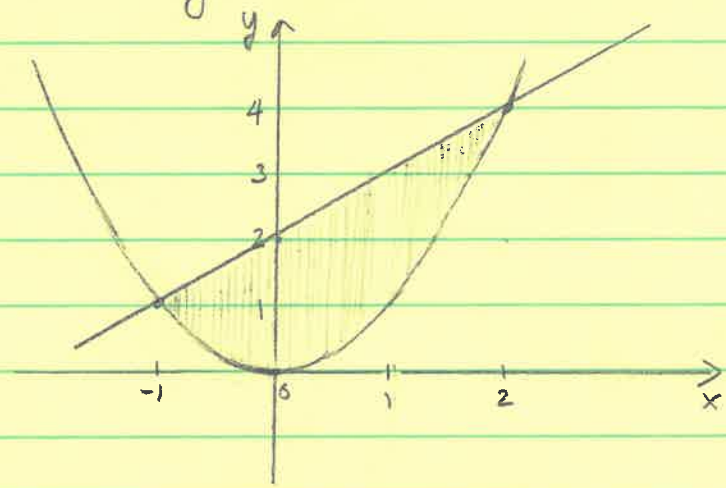
To find the exact area, we take the limit as $n \to \infty$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(c_i) - g(c_i) \right] \Delta x = \int_a^b \left[f(x) - g(x) \right] dx$$

When $f(x) \geq g(x)$ on $[a, b]$, the area between $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is given by

$$\boxed{A = \int_a^b \left[f(x) - g(x) \right] dx}$$

Ex. Find the area between $f(x) = x + 2$ and $g(x) = x^2$.



First we graph the curves. This lets us see the relationship between the curves.

We will also (in this case) determine a and b.

To find the points of intersection we can

1) read them off the graph and check them in the equations or
2) use algebra.

Curves intersect where $f(x) = g(x)$

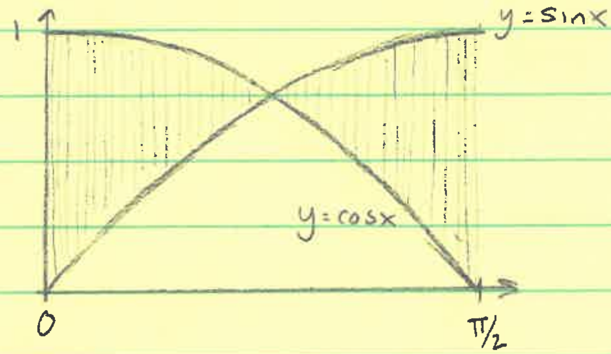
$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0 \implies x = -1, \; x = 2$$

so $a = -1$, $b = 2$

$$A = \int_{-1}^{2} \left[(x+2) - x^2\right] dx = \left.\frac{x^2}{2} + 2x - \frac{x^3}{3}\right|_{-1}^{2}$$
$$= \frac{4}{2} + 4 - \frac{8}{3} - \left[\frac{1}{2} - 2 + \frac{1}{3}\right]$$
$$= 6 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$
$$= 8 - \frac{1}{2} - 3$$
$$= 4.5$$

$\boxed{\text{Area is } 4.5 \text{ square units}}$

Ex. Find the area between $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \pi/2$



Which function plays the role of $f(x)$ (top)?

We can't just do $\int_0^{\pi/2} (\cos x - \sin x)\, dx$. Why not?

(point out that this integral is zero)

We need to split the interval into two parts.

$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \sin x + \cos x \Big|_0^{\pi/4} + -\cos x - \sin x \Big|_{\pi/4}^{\pi/2}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1) + [-0 - 1] - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} - 1 - 1 + \sqrt{2}$$

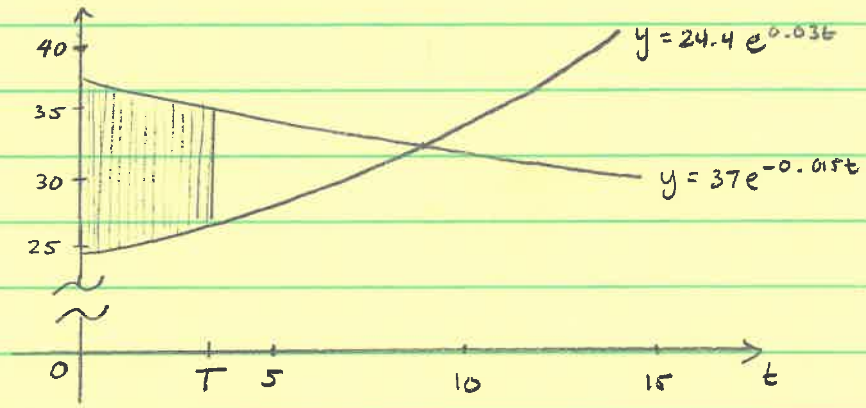
$$= 2\sqrt{2} - 2$$

$$= 2(\sqrt{2} - 1) \simeq 0.828427$$

Note: We need to carefully choose the order of functions in the integrand and the limits of integration.

Now, back to the reservoir problem.

1) Graphing $y = 37e^{-0.015t}$ and $y = 24.4e^{0.03t}$ we find



Notice that the area between $37e^{-0.015t}$ and $24.4e^{0.03t}$ from 0 to T is the net volume of water that enters the reservoir between $t = 0$ and $t = T$

Volume of Water in Reservoir

$$V = 12.8 + \int_0^T \left(37e^{-0.015t} - 24.4e^{0.03t}\right) dt$$

million gallons

Notice that this expression works even when $T > 10$ because we want the net amount of water to be entering the reservoir to be negative.

2) To find where V is maximized we can find the critical points of $V(t)$ and see if we have a maximum among them.

$$0 = \frac{dV}{dT} = \frac{d}{dT}\left[12.8 + \int_0^T (37e^{-0.015t} - 24.4e^{0.03t})\, dt\right]$$

$$0 = 0 + 37e^{-0.015T} - 24.4e^{0.03T}$$

by the Fundamental Theorem of Calculus

$$37e^{-0.015T} = 24.4e^{0.03T}$$

$$\frac{37}{24.4} = e^{0.03T+0.015T} = e^{0.045T}$$

$$0.045T = \ln\left(\frac{37}{24.4}\right)$$

$$T = \frac{1}{0.045}\ln\left(\frac{37}{24.4}\right)$$

$$T \simeq 9.2519 \text{ years}$$

This is the only critical point Is it a maximum?

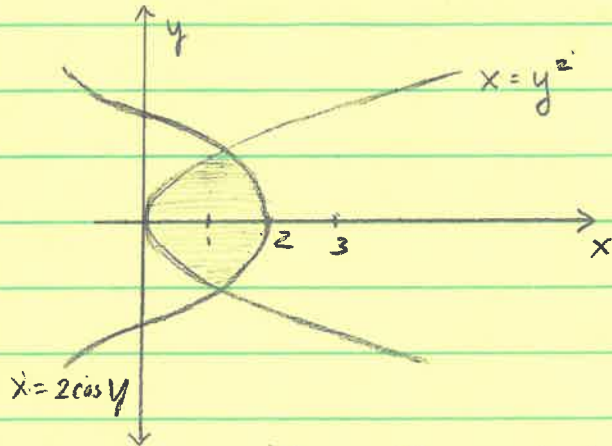
$$\frac{d^2V}{dT^2} = -0.015 \cdot 37e^{-0.015T} - 0.03 \cdot 24.4e^{0.03T}$$

$$= -0.555e^{-0.015T} - 0.732e^{0.03T} < 0 \quad \text{for all } T$$

Since $V'' < 0$ we know curve is concave down so the critical point must be a maximum.

Therefore, the amount of water in the reservoir is greatest after about 9.25 years.

Ex. Find the area between $x = y^2$ and $x = 2\cos y$



In this case x is a function y so we will be integrating along the y axis.

We will need to know the points where the curves intersect.

$$y^2 = 2\cos y$$

gives

$$y^2 - 2\cos y = 0$$

Using a calculator we find $y \simeq 1.02169$ and, by symmetry, $y = -1.02169$ are the y values of the points of intersection.

$$\therefore \quad \text{Area} = \int_{-1.02169}^{1.02169} (2\cos y - y^2)\,dy$$

or, by symmetry $\text{Area} = 2\int_0^{1.02169} (2\cos y - y^2)\,dy$

$$A = 2\left[2\sin y - \frac{y^3}{3}\right]_0^{1.02169}$$

$$\simeq 2.701$$

Area $\simeq 2.701$ square units