The Fundamental Theorem of Calculus

Why do we use such similar notation for antidifferentiation and computing definite integrals? The answer lies in our next theorem.

> **The Fundamental Theorem of Calculus I**
>
> If f is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof Let $[a,b]$ be partitioned into n intervals $[x_{i-1}, x_i]$ with $x_i - x_{i-1} = \Delta x$ where $\Delta x = \frac{b-a}{n}$ and $i = 1, 2, 3, \ldots, n$

Note that $x_0 = a$ and $x_n = b$.

$$F(b) - F(a) = F(x_n) - F(x_0)$$

$$= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots - F(x_2) + F(x_2) - F(x_1) + F(x_1) - F(x_0)$$

$$= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \cdots + [F(x_2) - F(x_1)] + [F(x_1) - F(x_0)]$$

$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$$ (Reversing the order of summation)

Since $F(x)$ is an antiderivative of $f(x)$, we know that F is differentiable on (a,b) and continuous on $[a,b]$. We can apply the MVT to find

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) = f(c_i)$$

for some $c_i \in [x_{i-1}, x_i]$.

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Rearranging, and using $\Delta x = x_i - x_{i-1}$, we have

$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x$$

So

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\Delta x$$

Taking the limit as $n \to \infty$ we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} F(b) - F(a)$$

$$\int_a^b f(c_i)\Delta x = F(b) - F(a) \quad \checkmark$$

Ex $\displaystyle\int_1^2 2x\,dx = 2^2 - 1^2$ since $\int 2x\,dx = x^2 + c$

Note two things

1. Since $F(x)$ can be any antiderivative of f, we can choose $c = 0$. In any event, any c will cancel when we form $F(b) - F(a)$.

2. We can use $F(x)\Big|_a^b = F(b) - F(a)$ to simplify our work.

Ex $$\int_0^\pi \sin x\,dx = (-\cos x)\Big|_0^\pi = -\cos\pi - (-\cos 0)$$
$$= -(-1) + 1$$
$$= 2.$$

Be very careful with signs!

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Ex $\int_1^2 \frac{1}{x}\,dx$

$$\int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2 - 0 = \boxed{\ln 2}$$

Ex $\int_1^x t\,dt$

Note that this is a definite integral, but one of the limits is a variable – therefore this is a function of x — change x and the function value changes.

$$\int_1^x t\,dt = \frac{1}{2}t^2\Big|_1^x = \frac{1}{2}x^2 - \frac{1}{2}(1)^2 = \frac{x^2}{2} - \frac{1}{2}$$

Note 2 things:

1) The variable of integration, t, is a "dummy" variable and must be different from the variable appearing in the limits of integration
2) This is an antiderivative of the function x (the integrand expressed with a different variable).

In general

$$\int_a^x t\,dt = \frac{1}{2}t^2\Big|_a^x = \frac{x^2}{2} - \frac{a^2}{2}$$

So the constant $-a^2/2$ is added to $x^2/2$, an antiderivative of x.

The Fundamental Theorem of Calculus II

If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ on $[a,b]$.

This can alternately be stated as

$$f(x) = \frac{d}{dx}\int_a^x f(t)\,dt$$

The Fundamental Theorem of Calculus

Ex If $F(x) = \int_3^x t\sin(t^2)\,dt$, find $F'(x)$.

$$f(x) = F'(x) = \frac{d}{dx}\int_3^x t\sin(t^2)\,dt$$

$$= \boxed{x\sin(x^2)}$$

Just substitute x for t in the integrand!

Q: Why does the lower limit of integration not matter?

Ex $\frac{d}{dx}\int_x^5 (t^2+t)\,dt$ Here x appears in the lower limit

Since $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$ [See eq 4.2 on page 376]

We can write

$$\frac{d}{dx}\int_x^5 (t^2+t)\,dt = \frac{d}{dx}\left[-\int_5^x (t^2+t)\,dt\right]$$

$$= \boxed{-(x^2+x)}$$

Ex $\frac{d}{dx}\int_0^{x^2} \sin t\,dt$. Let $u(x) = x^2$ so $\frac{du}{dx} = 2x$

$$\frac{d}{dx}\int_0^{x^2}\sin t\,dt = \frac{d}{dx}\int_0^u \sin t\,dt$$

$$= \left[\frac{d}{du}\int_0^u \sin t\,dt\right]\left[\frac{du}{dx}\right]$$

$$= (\sin u)(2x)$$

$$= \sin x^2 \cdot 2x = \boxed{2x\sin x^2}$$