Integration by Parts

Consider the integral $\int x e^{x^2} dx$. We can use the substitution $u = x^2$ to evaluate this.

Now consider $\int x e^x dx$. Substitution will not help with this integral. We can integrate x and e^x, but we don't know what to do with the product. Since this is a product, perhaps the product rule for differentiation will help.

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Solving for $f(x)g'(x)$ we have $f(x)g'(x) = \frac{d}{dx}\left[f(x)g(x)\right] - f'(x)g(x)$
Integrating wrt x gives

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Let $u = f(x)$ so $du = f'(x)\,dx$

$v = g(x)$ so $dv = g'(x)\,dx$

then

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The use of this formula is called <u>Integration by Parts</u>.

Ex $\int x e^x dx$ We need to identify the function u and a function for dv.

The function for u should "become simpler," or at least not get "more complicated" when we take its derivative.

The function for dv should not become "more complicated" when we find its antiderivative.

$u = x$ $\qquad dv = e^x dx$

$du = dx$ $\qquad v = e^x$

Integration by Parts

so $\int x e^x \, dx = \int u \, dv = uv - \int v \, du$

$= x e^x - \int e^x \, dx$ ← The key to integration by parts is that this integral should be simpler to evaluate than the original integral.

$= \boxed{x e^x - e^x + c}$

Ex $\int x^2 \cos x \, dx$ $\quad u = x^2 \quad dv = \cos x \, dx$

$du = 2x \, dx \quad v = \sin x$

$$\int x^2 \cos x \, dx = x^2 \sin x - \int \sin x \, (2x) \, dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

Well, we've gotten somewhere, but we still need integration by parts to evaluate this new integral

$u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$$= x^2 \sin x - 2 \left[-x \cos x + \int \cos x \, dx \right]$$

$$= \boxed{x^2 \sin x + 2x \cos x - 2 \sin x + c}$$

Check: $\frac{d}{dx}\left[x^2 \sin x + 2x \cos x - 2 \sin x + c\right]$

$= \cancel{2x \sin x} + x^2 \cos x + \cancel{2\cos x} - \cancel{2x \sin x} - \cancel{2 \cos x}$

$= x^2 \cos x$ ✓

Ex $\int x \ln x \, dx$ $\quad u = \ln x \quad dv = x \, dx$

$du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$

function for dv does get "more complicated" when integrated, but we don't yet know how to integrate $\ln x$ so we have no choice.

$$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c = \boxed{\frac{x^2}{4}(2 \ln x - 1) + c}$$

Integration by Parts

Ex $\displaystyle\int e^{ax}\sin bx\,dx$ $\qquad u=e^{ax}$, $du=ae^{ax}\,dx$ $\qquad dv=\sin bx\,dx$, $v=-\frac{1}{b}\cos bx$

$$= -\frac{e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

Notice that $\int e^{ax}\cos bx\,dx$ is "about the same" as the original integral.

Will using integration by parts work if we use it again? Let's try...

$$u=e^{ax} \qquad dv=\cos bx\,dx$$
$$du=ae^{ax} \qquad v=\frac{1}{b}\sin bx$$

$$I = -\frac{e^{ax}}{b}\cos bx + \frac{a}{b}\left[\frac{e^{ax}}{b}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx\right]$$

Hmm — We are back where we started; needing to evaluate $\int e^{ax}\sin bx\,dx$.

Let's use I to stand for the original integral so

$$I = -\frac{e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2}I + K$$ for some constant K (constant of integration)

and solve for I

$$I + \frac{a^2}{b^2}I = \frac{ae^{ax}}{b^2}\sin bx - \frac{e^{ax}}{b}\cos bx + K$$

multiply both sides by b^2 and factor out I on the LHS:

$$I(b^2+a^2) = ae^{ax}\sin bx - be^{ax}\cos bx + b^2K$$

$$I = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + c, \qquad c=\frac{b^2K}{a^2+b^2}$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + C}$$

<u>Warning</u>: Do not change your choices for u and dv during the second step — otherwise you end up with $0=0$.

Integration by Parts

Ex $\int \sin^n x \, dx$

$u = \sin^{n-1} x \qquad dv = \sin x \, dx$

$du = (n-1)\sin^{n-2} x \cdot \cos x \, dx \qquad v = -\cos x$

$$\int \sin^n x \, dx = -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \,(1-\sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$$

$$(1+n-1)\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx$$

$$\boxed{\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx}$$

This is an example of a <u>Reduction Formula</u>.

Ex $\int \sec^3 x\, dx$

This is a challenging problem, not because it is particularly difficult, but rather because it requires creativity and several steps

We begin by using integration by parts:

$$I = \int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$$

$u = \sec x$ $\qquad$ $dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx$ $\qquad$ $v = \tan x$

$$I = \sec x \tan x - \int \sec x \tan^2 x\, dx$$

Next, use $1 + \tan^2 x = \sec^2 x$ to get

$$I = \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

The first integral here is I, the one we are trying to evaluate, so we have

$$I = \sec x \tan x - I + \int \sec x\, dx$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x| + K$$

Dividing both sides by 2 gives

$$\boxed{\int \sec^3 x\, dx = \tfrac{1}{2} \sec x \tan x + \tfrac{1}{2} \ln|\sec x + \tan x| + C}$$