Other Applications of Integration

There are many other important applications to which integration can be applied. Here we consider a few.

As we talk about them look for the common theme: determine the appropriate expression for a changing quantity on a small interval, and then use integration to compute the sum (accumulation) of all contributing pieces.

Ex Work. We define work to be the product of the force being applied times the distance an object is moved.

- to lift 3 kg 250 cm (assuming gravity is 9.8 m/sec^2) we do

$$\underbrace{(3\text{ kg})(9.8\text{ m/sec}^2)}_{F=ma}(0.25\text{ m})$$

$$(29.4\text{ N})(0.25\text{m}) = 7.35\text{ Nm}$$
$$= 7.35\text{ J (joules)}$$

- Suppose water weighs 62.4 lb/ft^3. How much work is required to pump all the water out of a cylindrical tank with radius 2 ft and height 5 ft, assuming the tank is full and the water must be pumped to the top of the tank?

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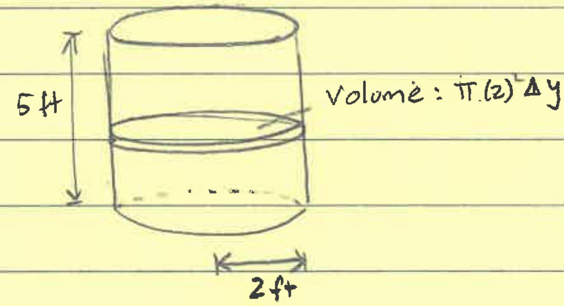
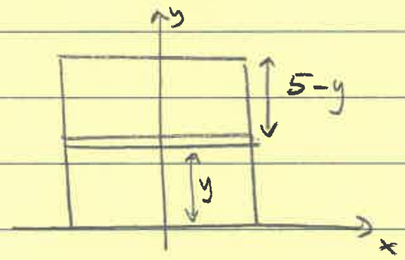


If we "divide" the water in the tank into "disks", then each disk must be raised a different height.



If the disk's height from the bottom of the tank is y, then the distance it must be raised is $5-y$.

- The force required to move the "disk" of water is $(62.4 \text{ lb/ft}^3)(\text{volume of disk ft}^3)$
- The volume of the disk is $\pi(2)^2 \Delta y = 4\pi \Delta y$
- Work to move "disk" at height y to top of tank:

$$(62.4 \text{ lb/ft}^3)(4\pi \Delta y \text{ ft}^3)(5-y) \text{ ft}$$
$$= 249.6\pi(5-y)\Delta y$$

- Approx work to empty tank

$$W \approx \sum_{i=1}^{n} 249.6\pi(5-y)\Delta y \qquad \text{where } \Delta y = \frac{5-0}{n}$$

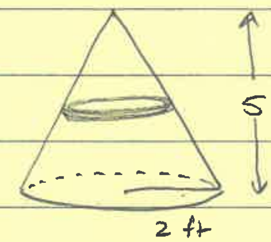
- Work to empty tank $= \lim_{n\to\infty} \sum_{i=1}^{n} 249.6\pi(5-y)\Delta y$

$$= \int_0^5 249.6\pi(5-y)\,dy$$

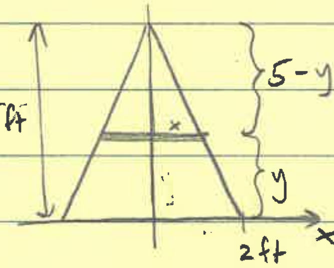
$$= 249.6\pi\left[5y - \tfrac{y^2}{2}\right]_0^5 = 249.6\pi\left(25 - \tfrac{25}{2}\right)$$

$$\boxed{= 3120\pi \text{ ft}\cdot\text{lbs}}$$

Now suppose the tank is conical, with base radius 2 ft and height 5 ft.



We proceed as before, but the size, and therefore weight, of each "disk" will depend on height.



- volume of disk : $\pi x^2 \, dy =$

$$\frac{x}{5-y} = \frac{2}{5} \rightarrow x = \frac{2}{5}(5-y)$$

$$\therefore \text{disk volume} = \pi \frac{4}{25}(5-y)^2 \, dy$$

- work to move "disk" to top of cone:

$$(62.4 \text{ lb/ft}^3)\left(\frac{4\pi}{25}(5-y)^2 dy \text{ ft}^3\right)(5-y \text{ ft})$$

- Work to empty tank:

$$W = \int_0^5 62.4 \frac{4\pi}{25}(5-y)^2(5-y)\,dy$$

$$= \frac{249.6\pi}{25}\int_0^5 (5-y)^3\,dy$$

$$= \frac{249.6\pi}{25}\int_0^5 (125 - 75y + 15y^2 - y^3)\,dy$$

$$= \frac{249.6\pi}{25}\left[125y - 75\frac{y^2}{2} + 5y^3 - \frac{y^4}{4}\right]_0^5$$

$$= 249.6\pi\left[25 - \frac{75}{2} + 25 - \frac{25}{4}\right]$$

$$= 249.6\pi\left[\frac{200}{4} - \frac{150}{4} - \frac{25}{4}\right] = 249.6\pi\frac{25}{4}$$

$$\boxed{= 1560\pi \text{ ft}\cdot\text{lbs}}$$

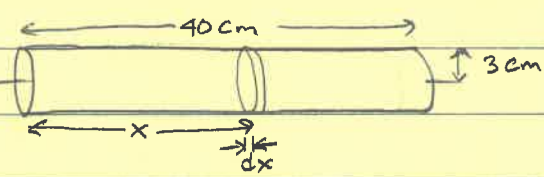
Other Applications of Integration

Ex Mass with varying density

Suppose one has a cylinder 6 cm in diameter and 40 cm long. The density of the material the cylinder is made of is 2.7 g/cm³. What is the mass of the cylinder?

$$\text{Mass} = \text{volume} \times \text{density}$$
$$= [\pi(3\text{cm})^2(40\text{cm})][2.7\ \text{g/cm}^3] = \boxed{972\pi\ \text{g}}$$
$$\approx \boxed{3053.6\ \text{g}}$$

Now suppose the density of the material varies along the length of the cylinder (eg. core sample) and is given by $\rho = 2 + x/10$ g/cm³ where x is the distance from one end of the cylinder.



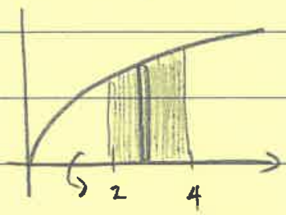
We can compute the mass of a thin "slice" (disk) of the cylinder: (volume of disk)(density at disk position)

$$= (\pi(3)^2 dx)(2 + x/10) = 9\pi(2 + x/10)\,dx \quad g$$

Integrating, to get the total mass, we have

$$M = \int_0^{40} 9\pi(2 + x/10)\,dx = 9\pi\left[2x + \frac{x^2}{20}\right]_0^{40} = 9\pi[80 + 80]$$
$$= \boxed{1440\pi\ g} \approx \boxed{4523.9\ g}$$

Find the mass of the solid produced when the region bounded by $y = \sqrt{x}$, the x-axis, and the lines $x = 2$ and $x = 4$ is rotated about the x-axis. Assume the density of the solid is given by $\rho = \frac{\ln x}{x^2}$



$$dv = \pi y^2 dx = \pi x \, dx$$

$$\text{mass of disk} = (\pi x \, dx)\left(\frac{\ln x}{x^2}\right) = \pi \frac{\ln x}{x} dx$$

$$M = \int_2^4 \pi \frac{\ln x}{x} dx \qquad u = \ln x, \quad du = \frac{1}{x} dx$$

$$u(2) = \ln 2, \quad u(4) = \ln 4 = 2\ln 2$$

$$= \int_{\ln 2}^{2\ln 2} \pi u \, du$$

$$= \pi \frac{u^2}{2} \Big|_{\ln 2}^{2\ln 2} = \frac{\pi}{2}\left[4(\ln 2)^2 - (\ln 2)^2\right] = \frac{3\pi}{2}(\ln 2)^2$$

$$= \boxed{\frac{3\pi}{2}(\ln 2)^2} \approx \boxed{2.2641}$$