Partial Fractions

Ex $\int \frac{2x+2}{x^2+2x+3}\,dx$ $\qquad$ $u = x^2+2x+3$, $du = (2x+2)\,dx$

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|x^2+2x+3| + C$$

Ex $\int \frac{1}{x^2+2x-3}\,dx$ $\qquad$ How can we proceed?

Completing the square?

$$x^2+2x-3 = x^2+2x+1-4$$
$$= (x+1)^2-4$$

so

$$\int \frac{1}{(x+1)^2-4}\,dx = \int \frac{1}{u^2-2^2}\,du \quad \text{when } u = x+1$$

Unfortunately this doesn't help... [can be done with $u = 2\sec\theta$?]

We notice, however, that we can factor the denominator

$$\int \frac{1}{x^2+2x-3}\,dx = \int \frac{1}{(x-1)(x+3)}\,dx$$

Suppose we knew $\frac{1}{(x-1)(x+3)} = \frac{1}{4(x-1)} - \frac{1}{4(x+3)}$. Then we have

$$\frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+3}\,dx$$

$$= \boxed{\frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C}$$

Ex Express $\frac{1}{x^2+x-6}$ as the sum of two rational functions

$$x^2+x-6 = (x-2)(x+3)$$

$$\frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

Where A and B are constants to be determined.

Multiplying by $(x-1)(x+3)$ to clear the denominators gives

$$1 = A(x+3) + B(x-2)$$

It is important to note that A and B are constants and that this equation must hold for all values of x.

We can easily find A and B by choosing $x=2$, $x=-3$.

When $x=2$ we have $1 = A(2+3) + B(0) \rightarrow 1 = 5A \rightarrow A = 1/5$

When $x=-3$ we have $1 = A(0) + B(-5) \rightarrow 1 = -5B \rightarrow B = -1/5$

So

$$\frac{1}{x^2+x-6} = \frac{1}{5(x-2)} - \frac{1}{5(x+3)}.$$

We call this operation a <u>partial fraction decomposition</u>.

We note the following

1) Every polynomial with real coefficients can be factored into linear and irreducible quadratic factors

2) Partial fractions can be used for $P(x)/Q(x)$ when $P(x)$ and $Q(x)$ are polynomials where the degree of P is <u>strictly</u> less than the degree of Q.

Ex Apply partial fractions to $\frac{3x-1}{(x-2)(x+1)(x+3)}$

$$\frac{3x-1}{(x-2)(x+1)(x+3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x+3}$$

Notice that the factors in the denominator are distinct.

To find A, B, C we again clear the fractions

$$3x-1 = A(x+1)(x+3) + B(x-2)(x+3) + C(x-2)(x+1)$$

This must be true for all values of x, but good ones to choose are $x=2$, $x=-1$, $x=-3$ (ask why!)

$x=2$: $\quad 5 = A(3)(5) \qquad 1 = 3A \qquad A = 1/3$

$x=-1$: $\quad -4 = B(-3)(2) \qquad -2 = -3B \qquad B = 2/3$

$x=-3$: $\quad -10 = C(-5)(-2) \qquad -10 = 10C \qquad C = -1$

So

$$\frac{3x-1}{(x-2)(x+1)(x+3)} = \frac{1}{3(x-2)} + \frac{2}{3(x+1)} - \frac{1}{x+3}$$

Rule: When $\dfrac{P(x)}{Q(x)} = \dfrac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)}$ where (a_ix+b_i)

are distinct factors of $Q(x)$ then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

Ex $\quad I = \displaystyle\int \frac{3x^3+32x^2+87x+72}{x^2+9x+14}\,dx$

Here the numerator has higher degree than the denominator.

In this case we first need to use long division.

Partial Fractions

$$\begin{array}{r|l} & 3x + 5 \\ x^2+9x+14 & 3x^3+32x^2+87x+72 \\ - & 3x^3+27x^2+42x \\ & \quad 5x^2+45x+72 \\ - & \quad 5x^2+45x+70 \\ & \qquad\qquad 2 \end{array}$$

So $$\frac{3x^3+32x^2+87x+72}{x^2+9x+14} = 3x+5 + \frac{2}{x^2+9x+14}$$

So
$$I = \int 3x+5 \, dx + \int \frac{2}{x^2+9x+14} \, dx$$

and we can use partial fractions on the second integral

$$\frac{2}{(x+2)(x+7)} = \frac{A}{(x+2)} + \frac{B}{(x+7)}$$

$$A = 2/5 \quad , \quad B = -2/5$$

$$I = 3\frac{x^2}{2} + 5x + \int \frac{2}{5}\frac{1}{x+2} \, dx - \int \frac{2}{5}\frac{1}{x+7} \, dx$$

$$= \boxed{\frac{3}{2}x^2 + 5x + \frac{2}{5}\ln|x+2| - \frac{2}{5}\ln|x+7| + C}$$

Partial Fractions

When linear factors are repeated in the denominator we need to proceed differently.

Rule: When $\dfrac{P(x)}{Q(x)} = \dfrac{P(x)}{(ax+b)^n}$ so that the denominator has n repeated factors, assume

$$\frac{P(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

Ex.

$$\frac{3x+1}{(x-1)(2x+1)^2} = \frac{A}{x-1} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2}$$

Clearing fractions we have

$$3x+1 = A(2x+1)^2 + B(x-1)(2x+1) + C(x-1)$$

When $x=1$ we have $4 = 9A$ so $A = 4/9$

When $x = -1/2$ we have $-1/2 = -\frac{3}{2}C$ so $C = 1/3$

Finding B requires a bit more work. One way is to move the known quantities to the LHS and simplify

$$3x+1 - \frac{4}{9}(2x+1)^2 - \frac{1}{3}(x-1) = B(x-1)(2x+1)$$

$$\frac{1}{9}\left[27x+9-4(4x^2+4x+1)-3x+3\right]$$

$$\frac{1}{9}\left[-16x^2+8x+8\right] = B(x-1)(2x+1)$$

$$-\frac{8}{9}\left[2x^2-x-1\right] = -\frac{8}{9}(2x+1)(x-1) = B(x-1)(2x+1)$$

So $B = -8/9$

$$\therefore \boxed{\frac{3x+1}{(x-1)(2x+1)^2} = \frac{4}{9}\frac{1}{x-1} - \frac{8}{9}\frac{1}{2x+1} + \frac{1}{3}\frac{1}{(2x+1)^2}}$$

Alternatively, start with

$$3x+1 = A(2x+1)^2 + B(x-1)(2x+1) + C(x-1)$$

$$= A[4x^2+4x+1] + B[2x^2-x-1] + C(x-1)$$

$$= 4Ax^2 + 4Ax + A + 2Bx^2 - Bx - B + Cx - C$$

$$3x+1 = (4A+2B)x^2 + (4A-B+C)x + (A-B-C)$$

For this to be true for all values of x, it must be that the coefficients of like powers of x on both sides of the equation must be equal.

$$4A+2B=0, \quad 4A-B+C=3, \quad A-B-C=1$$

$$B=-2A, \quad 4A+2A+C=3 \quad A+2A-C=1$$

$$6A+C=3 \quad 3A-C=1$$

$$C=3A-1$$

so $6A+3A-1=3$

$$9A=4 \rightarrow A=4/9$$

$$C=3\cdot 4/9-1 = 4/3-1 = 1/3$$

$$B=-8/9$$

so $$\boxed{\frac{3x+1}{(x-1)(2x+1)} = \frac{4}{9}\frac{1}{x-1} - \frac{8}{9}\frac{1}{2x+1} + \frac{1}{3}\frac{1}{(2x+1)^2}}$$

Partial Fractions

Finally, we consider the case where $Q(x)$ (the denominator) has irreducible quadratic factors

Rule: When $\dfrac{P(x)}{Q(x)} = \dfrac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_nx^2+b_nx+c_n)}$

where $a_ix^2+b_ix+c_i$ are irreducible quadratic factors of $Q(x)$ then

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{a_2x^2+b_2x+c_2} + \cdots + \frac{A_nx+B_n}{a_nx^2+b_nx+c_n}$$

Ex $$\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\begin{aligned} 1 &= Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 \\ &= Ax^3+Ax+Bx^2+B+Cx^3+Dx^2 \\ &= (A+C)x^3+(B+D)x^2+Ax+B \end{aligned}$$

$A+C=0$, $B+D=0$, $A=0$, $B=1$ so $A=C=0$
$B=1$, $D=-1$

$$\therefore \boxed{\frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}}$$

Ex. What should the form be for the partial fraction expansion of

$$\frac{x^2+2x-7}{(x^4-1)(x-1)(x+2)}$$

$$\frac{x^2+2x-7}{(x^2+1)(x+1)(x-1)^2(x+2)} = \boxed{\frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{x+2}}$$