Review of Formulas and Techniques

The focus in this section is on the basic techniques of integration developed in calculus I, including the use of the antiderivatives we developed in the last chapter.

There is a table summarizing the integrals we've developed so far at the start of Section 6.1. Today we'll work some examples that use these formulas.

Ex. $\int (3-x)^2\,dx$ We can either use substitution ($u = 3-x$) or expand the square

$$-\int u^2\,du = -\tfrac{1}{3}u^3 + C = \boxed{-\tfrac{1}{3}(3-x)^3 + C}$$

or $$\int 9 - 6x + x^2\,dx = \boxed{9x - 3x^2 + \tfrac{x^3}{3} + C}$$

These answers appear different, but we note that

$$-\tfrac{1}{3}(3-x)^3 = -\tfrac{1}{3}\left[9 - 27x + 9x^2 - x^3\right]$$
$$= -3 + 9x - 3x^2 + x^3/3$$

which matches our second answer, at least up to an additive constant.

Ex: $\int (3-x^2)^2\,dx$ In this integral substitution will not be of much help.

$$= \int (9 - 6x^2 + x^4)\,dx = \boxed{9x - 2x^3 + x^5/5 + C}$$

Ex $\int \frac{1}{a^2+x^2}\,dx$ This looks like $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$ except we have a^2 rather than 1 in the denominator

We can use some algebra to help out. What we want is a denominator that looks like $1 + (\ )^2$ so we need to divide both numerator and denominator by a^2.

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1/a^2}{1+x^2/a^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx$$

Now let $u = x/a$ so $du = \frac{1}{a}dx$ then we obtain

$$\frac{1}{a}\int \frac{1}{1+u^2}\,du = \frac{1}{a}\tan^{-1}u + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

so $\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c}$

Ex $\int \frac{1}{\sqrt{5-x^2}}\,dx$ We now try the same approach as in the last problem.

$$= \int \frac{1/\sqrt{5}}{\sqrt{\frac{5}{5}-\frac{x^2}{5}}}\,dx = \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-(x/\sqrt{5})^2}}\,dx$$

$u = x/\sqrt{5}$

$du = \frac{1}{\sqrt{5}}\,dx$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}u + C = \boxed{\sin^{-1}(x/\sqrt{5}) + C}$$

Ex $\int \frac{1}{x^2+4x+8}\,dx$ Substitution does not (immediately) work.

We proceed by <u>completing the square</u>.

$x^2+4x+8 = x^2+4x+4+4$ ⟵ what is left over

↑ take ½ of this and square it

$$= (x+2)^2+4$$

so $\int \frac{1}{(x+2)^2+4}\,dx = \int \frac{1}{2^2+(x+2)^2}\,dx$ Let $u = x+2$, $du = dx$

$$= \int \frac{1}{2^2+u^2}\,du = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)+c$$

$$= \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c}$$

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Ex $\int \sec x \, dx$

$$\int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1-\sin^2 x} du$$

$$u = \sin x \qquad du = \cos x \, dx$$

we get $\int \frac{1}{1-u^2} du$ Unfortunately we don't yet know how to deal with this yet...

We can, however, be clever...

$$\int \sec x \, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$u = \sec x + \tan x \qquad du = (\sec x \tan x + \sec^2 x) dx$$

so $\int \frac{1}{u} du = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$

Ex $\int \frac{1-x^2}{\sqrt{x}} dx = \int \left(\frac{1}{\sqrt{x}} - x^{3/2}\right) dx = \int x^{-1/2} dx - \int x^{3/2} dx$

$$= 2x^{1/2} - \tfrac{2}{5}x^{5/2} + C = 2\sqrt{x} - \tfrac{2}{5}x^2\sqrt{x} + C$$

$$= \boxed{2\sqrt{x}\left(1 - x^2/5\right) + C}$$