Sums, Riemann Sums and Area

$$1+2+3+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$2+4+6+\cdots+2n = \sum_{i=1}^{n} 2i$$

$$1+3+5+\cdots+47 = \sum_{i=1}^{24} (2i-1)$$

what does i need to be so $2i-1=47$?
$2i=48 \rightarrow i=24$

Thm: Let $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n} b_i$ be given. For any constants c and d

$$\sum_{i=1}^{n}(ca_i+db_i) = \sum_{i=1}^{n} ca_i + \sum_{i=1}^{n} db_i$$
$$= c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$$

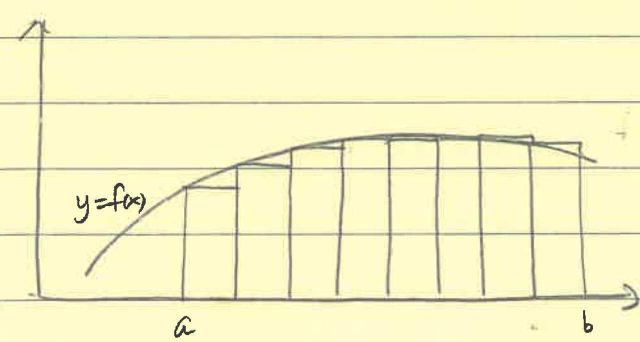
Ex $$\sum_{i=1}^{12}(3i^2-5i) = 3\sum_{i=1}^{12} i^2 - 5\sum_{i=1}^{12} i$$

$$= 3\frac{(12)(12+1)(2\cdot 12+1)}{6} - 5\frac{12(12+1)}{2}$$

$$= 6\cdot 13\cdot 25 - 30\cdot 13 = 150\cdot 13 - 30\cdot 13$$

$$= 120\cdot 13 = \underline{\underline{1560}}$$

Area under a curve can be approximated by multiple rectangles



The interval $[a,b]$ is partitioned into n pieces (usually of equal size) with endpoints $x_0, x_1, x_2, \ldots, x_n$.

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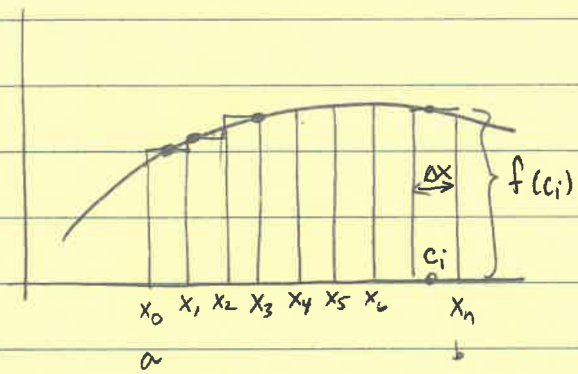

We determine the height of each rectangle by selecting an "evaluation point" $c_i \in [x_{i-1}, x_i]$. Note that c_i may be one of the sub-interval's endpoints.

To estimate the area "under" $f(x)$ from $x=a$ to $x=b$, we compute

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x$$

where $\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$ (notice in this case the sub intervals are all the same size)

Def: Let $\{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[a,b]$ with $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$ for $i = 1, 2, \ldots, n$. Let $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then the <u>Riemann sum</u> of $f(x)$ for this partition and set of evaluation points c_i is

$$\sum_{i=1}^{n} f(c_i)\Delta x .$$

Note: When $c_i = x_{i-1}$ we have the "Left" Riemann Sum

When $c_i = x_i$ we have the "Right" Riemann Sum

Ex Use summation to find the area under the curve $y = x^2$ on $[2, 8]$

$$\Delta x = \frac{8-2}{n} = \frac{6}{n} \quad , \quad x_i = 2 + \frac{6}{n} i$$

$$x_i^2 = \left(2 + \frac{6}{n} i\right)^2 = 4 + \frac{24}{n} i + \frac{36}{n^2} i^2$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 + \frac{24}{n} i + \frac{36}{n^2} i^2\right) \frac{6}{n}$$

$$= \lim_{n\to\infty} \frac{6}{n}\left[\sum_{i=1}^{n} 4 + \frac{24}{n}\sum_{i=1}^{n} i + \frac{36}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{6}{n}\left[4n + \frac{24}{n}\frac{n(n+1)}{2} + \frac{36}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{6}{n}\left[4n + \frac{12(n^2+n)}{n} + \frac{6(2n^3+3n^2+n)}{n^2}\right]$$

$$= \lim_{n\to\infty} \left[24 + 72\frac{n^2+n}{n^2} + 36\frac{2n^3+3n^2+n}{n^3}\right]$$

$$= 24 + 72 + 72 = 168$$

$\boxed{\text{Area} = 168}$