Vectors in the Plane and in Space

What is the difference between velocity and speed?

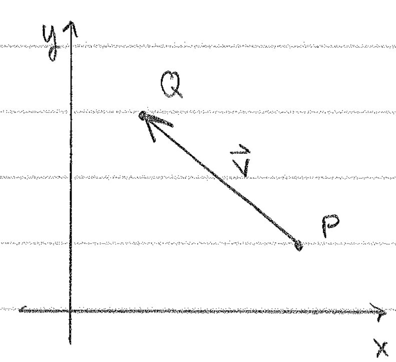
If $s(t)$ gives the position of an object then ds/dt gives the velocity and $|ds/dt|$ is the speed.

$ds/dt >$ means object is moving to the right; $\frac{ds}{dt} < 0$ means the object is moving to the left. $\frac{ds}{dt} = 0$ means the object is not moving (stationary)

So far in calculus we have only considered functions of one variable — we now want to extend this to functions of more than one variable.

In the case of motion, this means that we need more direction information than "left" or "right"

The key tool for this is a vector, which we will define as a quantity with both magnitude (size) and direction.



$\vec{v} = \overrightarrow{PQ}$ the directed line segment from P to Q.

Vectors have magnitude and direction but are independent of position

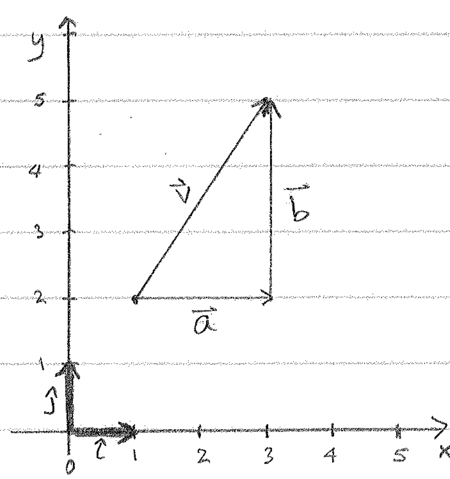
Often we draw vectors with their tail at the origin. These are called position vectors. The head of a position vector locates a point.

Vectors, as we have defined them, are not restricted to the plane, but also exist in space (3 dimensions) and even higher dimensions.

How can we specify a vector? ① graphically - but this has limitations, ② By using an agreed upon set of vectors as building blocks, and ③ by specifying the point located by a position vector.

We often use the coordinate unit vectors $\hat{i}$ and $\hat{j}$ in the plane or $\hat{i}, \hat{j}$, and $\hat{k}$ in space.

- unit vectors have length 1
- $\hat{i}$ is parallel to the x-axis, $\hat{j}$ is parallel to the y-axis, $\hat{k}$ is parallel to the z-axis



If we agree to define the addition of two vectors by placing one vector's tail on the other's head and let the sum be the vector from the tail of the first vector to the head of the second then $\vec{v} = \vec{a} + \vec{b}$.

Further, if we define the product of a scalar c and a vector to be a new vector parallel to the first but with a length scaled by c, then we can see $\vec{a} = 2\hat{\imath}$ and $\vec{b} = 3\hat{\jmath}$

$$\therefore \ \vec{V} = 2\hat{\imath} + 3\hat{\jmath}$$

We can also write $\vec{V} = \langle 2, 3 \rangle$. In general, if $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then $\vec{V} = \langle x, y, z \rangle$. Here x, y, and z are called the components of $\vec{V}$.

Notice:

1. $\vec{V} = 2\hat{\imath} + 3\hat{\jmath}$ can have <u>any location</u> but will always be parallel to the position vector from the origin to the point (2,3)
2. Vectors and scalars are different! Be sure what type of quantity you are working with
3. We write $\vec{V}$ with an arrow to show $\vec{V}$ is a vector; our text uses <u>bold</u>.

The <u>magnitude</u> of a vector $\vec{V} = \langle x, y \rangle = x\hat{\imath} + y\hat{\jmath}$ is defined to be

$$\|\vec{V}\| = \sqrt{x^2 + y^2}$$

Similarly, if $\vec{V} = \langle x, y, z \rangle = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then

$$\|\vec{V}\| = \sqrt{x^2 + y^2 + z^2}$$

The set of all vectors in the plane is V_2 in our book, elsewhere it is denoted $\mathbb{R}^2$. The set of vectors in space is V_3 (or $\mathbb{R}^3$).

Theorem: For any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ in V_2, and any scalars d and e in $\mathbb{R}$, the following hold

1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutativity)
2) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$ (associativity)
3) $\vec{a}+\vec{0}=\vec{a}$ (additive identity)
4) $\vec{a}+(-\vec{a})=\vec{0}$ (additive inverse)
5) $d(\vec{a}+\vec{b})=d\vec{a}+d\vec{b}$ (distributive law)
6) $(d+e)\vec{a}=d\vec{a}+e\vec{a}$ (distributive law)
7) $1\vec{a}=\vec{a}$ (mult. by one)
8) $0\vec{a}=\vec{0}$ (mult by zero)

NOTE: - $\vec{0}$ is the <u>zero vector</u>, has no direction and zero magnitude
- V_2 can be replaced by V_3 to extend these properties to space.

Examples

① Add $\vec{u}=2\hat{\imath}+3\hat{\jmath}$ to $\vec{v}=\hat{\imath}-5\hat{\jmath}$

$$\vec{u}+\vec{v}=2\hat{\imath}+3\hat{\jmath}+\hat{\imath}-5\hat{\jmath}=(2\hat{\imath}+\hat{\imath})+(3\hat{\jmath}-5\hat{\jmath})=3\hat{\imath}-2\hat{\jmath}$$

② Compute $3(\hat{\imath}-2\hat{\jmath})+4(3\hat{\imath}+2\hat{\jmath})$

$$=3\hat{\imath}-6\hat{\jmath}+12\hat{\imath}+8\hat{\jmath}=3\hat{\imath}+12\hat{\imath}+(-6\hat{\jmath}+8\hat{\jmath})=15\hat{\imath}+2\hat{\jmath}$$

Examples

③ Let $\vec{v} = 3\hat{\imath} + 4\hat{\jmath}$ and find a <u>unit vector</u> (vector of length 1) parallel to $\vec{v}$.

We can always find a unit vector in the direction of $\vec{v}$ by $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$. So, $\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$\boxed{\hat{v} = \frac{3\hat{\imath}+4\hat{\jmath}}{5} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}.}$ Another answer is $-\hat{v}$ or $-\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$.

④ Let $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$, $\vec{b} = 4\hat{\imath} - 3\hat{k}$. Compute $\|-\vec{a} + 2\vec{b}\|$.

$$-\vec{a} + 2\vec{b} = -(\hat{\imath}+\hat{\jmath}-\hat{k}) + 2(4\hat{\imath}-3\hat{k}) = -\hat{\imath}-\hat{\jmath}+\hat{k}+8\hat{\imath}-6\hat{k}$$
$$= 7\hat{\imath} - \hat{\jmath} - 5\hat{k}$$
$$\therefore \|-\vec{a}+2\vec{b}\| = \sqrt{7^2+(-1)^2+(-5)^2} = \sqrt{49+1+25} = \sqrt{75}$$

⑤ Find the vector $\vec{a}$ so $6\hat{\imath} - 7\hat{\jmath} + \vec{a} = -2\hat{\jmath}$

$$\begin{array}{rcl} 6\hat{\imath}-7\hat{\jmath}+\vec{a} & = & -2\hat{\jmath} \\ -(6\hat{\imath}-7\hat{\jmath}) & & -(6\hat{\imath}-7\hat{\jmath}) \\ \hline \vec{a} & = & -2\hat{\jmath}-6\hat{\imath}+7\hat{\jmath} = -6\hat{\imath}+5\hat{\jmath} \end{array}$$

$$\boxed{\vec{a} = -6\hat{\imath}+5\hat{\jmath}}$$

⑥ Find the vector from $(1,3,-2)$ to $(7,0,1)$

$$(7-1)\hat{\imath} + (0-3)\hat{\jmath} + (1-(-2))\hat{k} = \boxed{6\hat{\imath}-3\hat{\jmath}+3\hat{k}}$$