The Dot Product

Def: The dot product of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ in V_3 is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

If $\vec{a}$ and $\vec{b}$ are in V_2 then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

Ex $\quad (3\hat{\imath} + 2\hat{\jmath} - 5\hat{k}) \cdot (\hat{\jmath} - 3\hat{k}) = 3 \cdot 0 + 2 \cdot 1 + (-5)(-3) = 2 + 15 = 17$

Thm: For vectors $\vec{a}, \vec{b}$ and $\vec{c}$ and any scalar d,

i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

ii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

iii) $(d\vec{a}) \cdot \vec{b} = d(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (d\vec{b})$

iv) $\vec{0} \cdot \vec{a} = 0$

v) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

Proof: We will assume vectors are in V_3 — the adjustment to V_2 is straightforward

i) see text

ii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$

$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$$

$$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 c_1 + a_2 c_2 + a_3 c_3$$

$$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

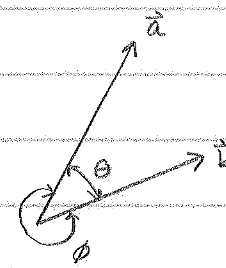
iii) $(d\vec{a}) \cdot \vec{b} = \langle da_1, da_2, da_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle$

$$= da_1 b_1 + da_2 b_2 + da_3 b_3$$

$$= d(a_1 b_1 + a_2 b_2 + a_3 b_3) = d(\vec{a} \cdot \vec{b})$$

iv) $\vec{0}\cdot\vec{a} = 0\cdot a_1 + 0\cdot a_2 + 0\cdot a_3 = 0$

v) see text.



The angle between two vectors is the smaller of the two angles formed when the initial points (tails) of the two vectors are placed together. In this case it is θ, not ϕ.
This angle will always satisfy $0 \le \theta \le \pi$

Two vectors are said to be orthogonal if the angle between them is $\pi/2$ (i.e. they are perpendicular)

Thm: Let θ be the angle between two nonzero vectors $\vec{a}$ and $\vec{b}$.
Then

$$\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta}$$

Proof: (by cases)

1) If $\theta = 0$ then $\vec{b} = c\vec{a}$ for some $c > 0$.

$$\vec{a}\cdot\vec{b} = c\,\vec{a}\cdot\vec{a} = c\|\vec{a}\|^2$$

same ✓

$$\|\vec{a}\|\,\|\vec{b}\|\cos\theta = \|\vec{a}\|\,c\|\vec{a}\|\cos 0 = c\|\vec{a}\|^2$$

2) If $\theta = \pi$ then $\vec{b} = c\vec{a}$ for some $c < 0$

$$\vec{a}\cdot\vec{b} = c\,\vec{a}\cdot\vec{a} = c\|\vec{a}\|^2$$

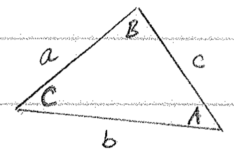
Same ✓

$$\|\vec{a}\|\,\|\vec{b}\|\cos\theta = \|\vec{a}\|\,|c|\,\|\vec{a}\|\cos\pi$$

$$= (-c)\|\vec{a}\|^2(-1) = c\|\vec{a}\|^2$$

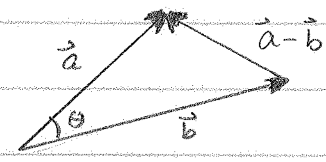
The Dot Product

3) If $0 < \theta < \pi$ we will use the Law of Cosines



$c^2 = a^2 + b^2 - 2ab\cos C$

So, we have



$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

but $\|\vec{a}-\vec{b}\|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$$= \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$$= \|\vec{a}\|^2 - 2\,\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$

$\therefore$ $\|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\,\vec{a}\cdot\vec{b} = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\|\cos\theta$

So $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$ ✓

Ex Find the angle between $\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{j} - 3\hat{k}$

$\vec{a}\cdot\vec{b} = 17$ (from page 1 of notes)

$\|\vec{a}\| = \sqrt{3^2 + 2^2 + (-5)^2} = \sqrt{9+4+25} = \sqrt{38}$

$\|\vec{b}\| = \sqrt{0^2 + 1^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$

$$17 = \sqrt{38}\,\sqrt{10}\,\cos\theta$$

$$\cos\theta = \frac{17}{\sqrt{38}\,\sqrt{10}}$$

$$\theta = \cos^{-1}\left(\frac{17}{\sqrt{38}\,\sqrt{10}}\right) \approx 0.51136 \text{ radians}$$

$$\approx 29.3^\circ$$

The Dot Product

The vectors $\vec{a}$ and $\vec{b}$ are orthogonal when the angle between them is $\pi/2$. Therefore

$$\vec{a}\cdot\vec{b}=0 \quad \text{iff} \quad \vec{a} \text{ is orthogonal to } \vec{b}$$

Ex. Find x so that $\vec{a}=x\hat{\imath}+4\hat{\jmath}$ is orthogonal to $\vec{b}=2\hat{\imath}-5\hat{\jmath}$.

$$\vec{a}\cdot\vec{b}=2x+4(-5)=0$$

$$2x-20=0$$

$$x=10 \rightarrow \vec{a}=10\hat{\imath}+4\hat{\jmath}$$

Thm: Cauchy-Schwartz Inequality

For any vectors $\vec{a}$ and $\vec{b}$

$$\boxed{|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|}$$

Proof: If $\vec{a}$ or $\vec{b}$ (or both) is the zero vector then this is trivially true since $0 \leq 0$.

Otherwise, $|\vec{a}\cdot\vec{b}| = \big|\|\vec{a}\|\|\vec{b}\|\cos\theta\big| = \|\vec{a}\|\|\vec{b}\||\cos\theta| \leq \|\vec{a}\|\|\vec{b}\|$

Since $|\cos\theta| \leq 1$

Thm: Triangle Inequality

For any vectors $\vec{a}$ and $\vec{b}$

$$\boxed{\|\vec{a}+\vec{b}\| \leq \|\vec{a}\|+\|\vec{b}\|}$$

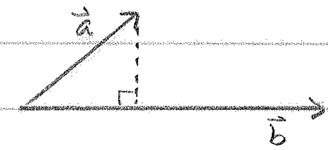
Proof: $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}+2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b} = \|\vec{a}\|^2+2\vec{a}\cdot\vec{b}+\|\vec{b}\|^2$

$\leq \|\vec{a}\|^2+2|\vec{a}\cdot\vec{b}|+\|\vec{b}\|^2 \leq \|\vec{a}\|^2+2\|\vec{a}\|\|\vec{b}\|+\|\vec{b}\|^2 = (\|\vec{a}\|+\|\vec{b}\|)^2$

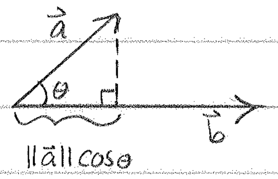
$\therefore \|\vec{a}+\vec{b}\| \leq \|\vec{a}\|+\|\vec{b}\|$ ✓

The Dot Product

In many cases we want to know the component of one vector in the direction of another vector.



We can work this out by "dropping a perpendicular line segment" from the terminal point of $\vec{a}$ to the line containing the vector $\vec{b}$.



The corresponding length along $\vec{b}$ is $\|\vec{a}\| \cos\theta$ which is $\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$.

$$\boxed{\operatorname{Comp}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}}$$

is the component of $\vec{a}$ in the direction of $\vec{b}$

Note that $\operatorname{Comp}_{\vec{b}}\vec{a}$ is a <u>scalar quantity</u>!

To find the <u>projection of $\vec{a}$ onto $\vec{b}$</u> we need to multiply the scalar component value by a unit vector in the direction of $\vec{b}$.

$$\operatorname{Proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\,\frac{\vec{b}}{\|\vec{b}\|} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b}$$

$$\boxed{\operatorname{Proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}}$$

This is a vector quantity!

Ex. Find the projection of $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} - 5\hat{k}$ onto $\vec{b} = \hat{\jmath} - 3\hat{k}$

$$\operatorname{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b} = \frac{17}{0+1^2+(-3)^2}(\hat{\jmath}-3\hat{k}) = \frac{17}{10}(\hat{\jmath}-3\hat{k}) = \underline{1.7\hat{\jmath} - 5.1\hat{k}}$$