# The Cross Product



Def The 2x2 (read "2 by 2") determinant is computed

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The 3x3 determinant is computed

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

note this minus sign

This is called "co-factor expansion about the first row".

Def For two vectors $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ in $V_3$, we define the cross product of $\vec{a}$ and $\vec{b}$ to be

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

* The cross product is a vector in $V_3$ *

Ex $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$, $\vec{b} = \hat{\imath} + 3\hat{\jmath} + 5\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 3 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 3 & 5 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \hat{k}$$

$$= (-5-9)\hat{\imath} - (10-3)\hat{\jmath} + (6+1)\hat{k}$$

$$= -14\hat{\imath} - 7\hat{\jmath} + 7\hat{k}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 5 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ -1 & 3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \hat{k}$$

$$= (9+5)\hat{\imath} - (3-10)\hat{\jmath} + (-1-6)\hat{k}$$

$$= 14\hat{\imath} + 7\hat{\jmath} - 7\hat{k}$$

# The Cross Product



Thm: For any vector $\vec{a} \in V_3$, $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times \vec{0} = \vec{0}$

Proof: $$\vec{a} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} \hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$\vec{a} \times \vec{0} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix} \hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

We know that $\vec{a} \times \vec{b}$ is a vector. Can we determine anything about its direction relative to the directions of $\vec{a}$ & $\vec{b}$?

Let's compute $\vec{a} \cdot (\vec{a} \times \vec{b})$

$$\begin{aligned} \vec{a} \cdot (\vec{a} \times \vec{b}) &= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \cdot \left[ (a_2b_3 - a_3b_2)\hat{\imath} - (a_1b_3 - a_3b_1)\hat{\jmath} + (a_1b_2 - a_2b_1)\hat{k} \right] \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0 \end{aligned}$$

A similar thing happens for $\vec{b} \cdot (\vec{a} \times \vec{b})$

If the dot product of two vectors is zero then the vectors are orthogonal.

Thm: For any two vectors $\vec{a}$ and $\vec{b}$ in $V_3$, $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

We now know that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$, but in which of the two possible directions does it point?

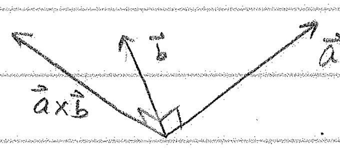
# The Cross Product



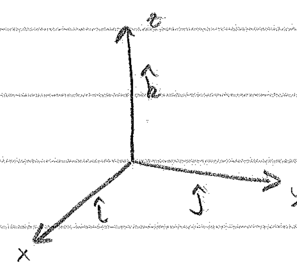
The cross product obeys the "right hand rule":

Using your right hand

1) point your fingers in the direction of $\vec{a}$
2) "curl" your fingers into the direction of $\vec{b}$
3) your extended thumb will point in the direction of $\vec{a}\times\vec{b}$




What does this rule say about the directions of $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$?




What is $\hat{\imath}\times\hat{\jmath} = \hat{k}$ $\qquad \hat{\jmath}\times\hat{\imath} = -\hat{k}$

$\hat{\jmath}\times\hat{k} = \hat{\imath} \qquad \hat{k}\times\hat{\jmath} = -\hat{\imath}$

$\hat{k}\times\hat{\imath} = \hat{\jmath} \qquad \hat{\imath}\times\hat{k} = -\hat{\jmath}$

Ex $\left.\begin{aligned}(\hat{\imath}\times\hat{\jmath})\times\hat{\jmath} &= \hat{k}\times\hat{\jmath} = -\hat{\imath}\\ \hat{\imath}\times(\hat{\jmath}\times\hat{\jmath}) &= \hat{\imath}\times\vec{0} = \vec{0}\end{aligned}\right\}$ so $(\hat{\imath}\times\hat{\jmath})\times\hat{\jmath} \neq \hat{\imath}\times(\hat{\jmath}\times\hat{\jmath})$

not associative

Thm: For any vectors $\vec{a},\vec{b},\vec{c}\in V_3$ and $d\in\mathbb{R}$

i) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

ii) $(d\vec{a})\times\vec{b} = d(\vec{a}\times\vec{b}) = \vec{a}\times(d\vec{b})$

iii) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

iv) $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

v) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

vi) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

## The Cross Product



Recall that $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$, $0 \le \theta \le \pi$ so

- $\vec{a}\cdot\vec{b}$ is as positive as possible when $\vec{a}$ and $\vec{b}$ point in the same direction
- $\vec{a}\cdot\vec{b}$ is zero if $\vec{a}$ and $\vec{b}$ are orthogonal
- as negative as possible when $\vec{a}$ and $\vec{b}$ point in opposite directions

Now examine $\|\vec{a}\times\vec{b}\|$.

$$\|\vec{a}\times\vec{b}\|^2 = (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2$$

$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \quad \boxed{\text{See text}}$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta)$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$

$$\therefore \quad \|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$$

**Thm:** For nonzero vectors $\vec{a}$ and $\vec{b}$ in $V_3$, if $\theta$ is the angle $(0 \le \theta \le \pi)$ between $\vec{a}$ and $\vec{b}$ then

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta$$

and $$\vec{a}\times\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\sin\theta\ \hat{n}$$

where $\hat{n}$ is a unit vector whose direction is determined by the right hand rule.

**Corollary:** Two nonzero vectors $\vec{a}$ and $\vec{b}$ in $V_3$ are parallel if and only if $\vec{a}\times\vec{b} = \vec{0}$