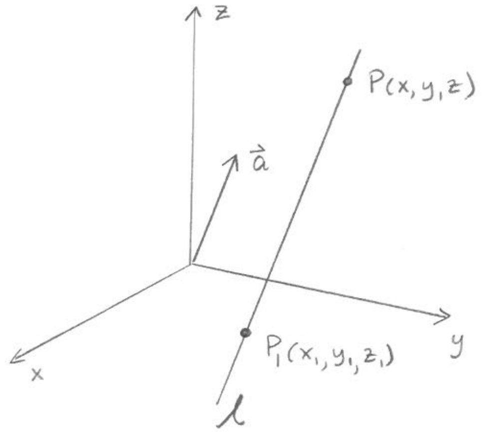
Lines and Planes in Space



Suppose we want to find the equation of a line in space. We need either

1) Two points on the line, or
2) a point on the line and a direction (i.e. a vector)

Consider the line ℓ passing through point P_1 and parallel to $\vec{a}$.

$$\overrightarrow{P_1P} \parallel \vec{a} \quad \text{so} \quad \overrightarrow{P_1P} = t\vec{a} \text{ for some } t \in \mathbb{R}$$

but

$$\overrightarrow{P_1P} = (x-x_1)\hat{\imath} + (y-y_1)\hat{\jmath} + (z-z_1)\hat{k}$$

so

$$(x-x_1)\hat{\imath} + (y-y_1)\hat{\jmath} + (z-z_1)\hat{k} = ta_1\hat{\imath} + ta_2\hat{\jmath} + ta_3\hat{k}$$

$\therefore$

$$\boxed{\begin{array}{lcl} x - x_1 = a_1 t & \quad\text{or}\quad & x = x_1 + a_1 t \\ y - y_1 = a_2 t & & y = y_1 + a_2 t \\ z - z_1 = a_3 t & & z = z_1 + a_3 t \end{array}}$$

These are the parametric equations of the line ℓ.
t is a parameter.

Notice that as t increases from zero P moves away from P_1 in the direction of $\vec{a}$ while if t decreases from zero then P moves away from P_1 in the opposite direction

Solving each parametric equation for t and equating gives

$$\boxed{\frac{x-x_1}{a_1} = \frac{y-y_1}{a_2} = \frac{z-z_1}{a_3}}$$

whenever none of a_1, a_2, a_3 are zero. These are the symmetric equations of the line.

Ex Find the equations of a line parallel to $\vec{a} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ and passing through the point $(0, 1, -3)$.

Here $\langle a_1, a_2, a_3 \rangle = \langle 3, -2, 5 \rangle$ and $(x_1, y_1, z_1) = (0, 1, -3)$ so

$$x = 3t \quad , \quad y = 1 - 2t, \quad z = -3 + 5t$$

are the parametric equations and

$$\frac{x}{3} = \frac{y-1}{-2} = \frac{z+3}{5}$$

are the symmetric equations.

Ex Find the point where the line in the last example intersects the xy-plane.

Since $z = 0$ for all points on the xy-plane, we can use the symmetric equations to find

$$\frac{x}{3} = \frac{y-1}{-2} = \frac{3}{5} \rightarrow x = \frac{9}{5},$$

and $y - 1 = \frac{-6}{5} \rightarrow y = -\frac{1}{5}$

$\therefore$ The point of intersection is $\left(\frac{9}{5}, -\frac{1}{5}, 0\right)$

In the plane, given any two lines they will either be parallel or they will intersect.

This is not true in Space.

Ex Show the lines given by

① $x = 1+t, \quad y = 2-3t, \quad z = 7-t$ and

② $x = 3-s, \quad y = -s, \quad z = 5+2s$

will not intersect and are not parallel.

We begin by finding vectors parallel to the lines.

$x - x_1 = a_1 t \Rightarrow x - 1 = t$ and $x - 3 = (-1)s$

$y - y_1 = a_2 t \Rightarrow y - 2 = -3t$ and $y - 0 = (-1)s$

$z - z_1 = a_3 t \Rightarrow z - 7 = (-1)t$ and $z - 5 = 2s$

$\vec{a} = \hat{\imath} - 3\hat{\jmath} - \hat{k} \qquad \hat{b} = -\hat{\imath} - \hat{\jmath} + 2\hat{k}$

It is easy to see $\vec{a}$ is not parallel to $\vec{b}$ since it is not a scalar multiple of b. Therefore the lines are not parallel.

(could also show $\|\vec{a} \times \vec{b}\| \neq 0$)

Equating $x = 1+t = 3-s$ and $y = 2-3t = -s$, we can solve for t and s.

Since $-s = 2-3t$, $1+t = 3-s \Rightarrow 1+t = 3+2-3t$

$4t = 4 \Rightarrow t = 1$

If $t=1$ then $s=1$.

In this case

$z = 7-t = 7-1 = 6$

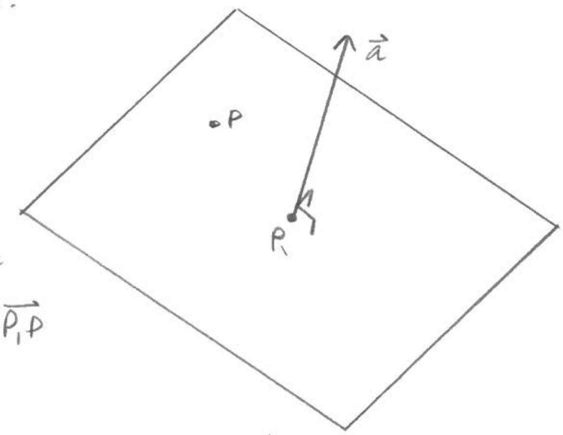
and $z = 5+2s = 5+2 = 7$

Since these are not equal, the lines do not intersect.

Nonintersecting, nonparallel lines are called **skew** lines.

Planes

We've just seen that a point and a vector is enough to define a line in space. It turns out that they are also enough to define a plane.



Again we use the vector $\overrightarrow{P_1P}$ but now we want to describe the set of points P so that $\overrightarrow{P_1P}$ is orthogonal to $\vec{a}$.

$$\overrightarrow{P_1P} = (x-x_1)\hat{\imath} + (y-y_1)\hat{\jmath} + (z-z_1)\hat{k}$$

so

$$\vec{a}\cdot\overrightarrow{P_1P} = a_1(x-x_1) + a_2(y-y_1) + a_3(z-z_1) = 0$$

$$a_1x + a_2y + a_3z - (a_1x_1 + a_2y_1 + a_3z_1) = 0$$

This is the equation of the plane.

The general plane equation is sometimes written $Ax + By + Cz + D = 0$
here $A = a_1$, $B = a_2$, $C = a_3$, $D = -a_1x_1 - a_2y_1 - a_3z_1$

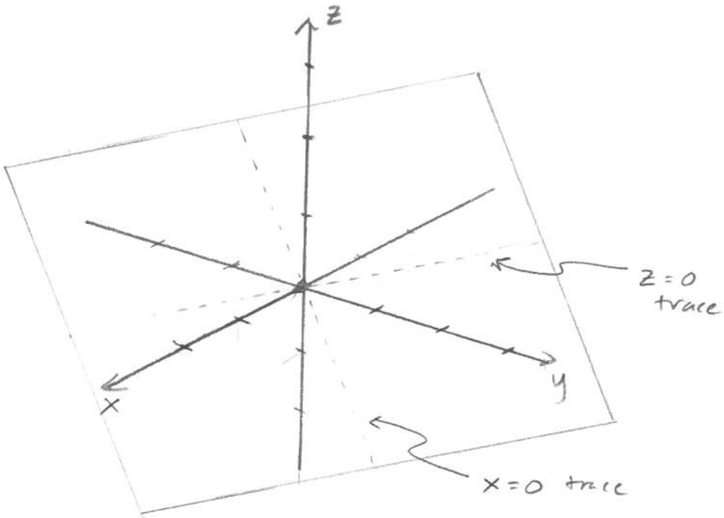
Ex Find the equation of the plane through the origin and orthogonal to the vector $\vec{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$. Sketch the plane.

The vector $\vec{a}$ is normal to the plane, and its components are the coefficients in the plane equation

$$1x + 2y + 1z = d$$

where d is chosen so $(0,0,0)$ is in the plane. $\therefore\ d = 0$

$\therefore \quad x + 2y + z = 0$ is the equation of the plane.



To graph the plane, we

1. locate a point on the plane – in this case, the origin since $(0,0,0)$ satisfies our equation.
2. Fix one variable and construct a trace. Then repeat with another variable. Here we used $x=0$ and $y=0$
3. Draw segment of plane with edges parallel to traces.

<u>Ex</u> Find the line of intersection between the planes

$x - 2y + z = 5$ and $4x + y + 2z = -2$

The key idea is that the same set of points should satisfy both plane equations. Solving both equations for z gives

$z = 5 - x + 2y \qquad z = -1 - 2x - \frac{1}{2}y$

$5 - x + 2y = -1 - 2x - \frac{1}{2}y$

$6 + x + \frac{5}{2}y = 0$

$\frac{5}{2}y = -6 - x \rightarrow y = -\frac{2}{5}x - \frac{12}{5}$ [y in terms of x]

$z = 5 - x + 2\left(-\frac{2x}{5} - \frac{12}{5}\right) = 5 - x - \frac{4x}{5} - \frac{24}{5}$

$z = -\frac{9x}{5} + \frac{1}{5}$ [z in terms of x]

Let $x = 5t$

$y = -\frac{12}{5} - 2t$

$z = \frac{1}{5} - 9t$