Functions of Several Variables

__Def__ A function of two variables is a rule that assigns a real number $z = f(x,y)$ to each ordered pair of real numbers (x,y) in the domain of the function.

$f: D \to \mathbb{R}$ where $D \subset \mathbb{R}^2$

__Ex__ $f(x,y) = xy$

$f(x,y) = x^2 + y^2$

$f(x,y) = 1 - x + e^y$

The definition can be extended to a function of n variables in an obvious way.

__Ex__ Determine the domain of the following functions

1) $f(x,y) = y/x$ Defined as long as $x \neq 0$.

$\boxed{D = \{(x,y) \mid x \neq 0\}}$

2) $f(x,y) = \dfrac{xy}{x^2 - y^2}$ Defined as long as $x^2 - y^2 \neq 0$

So $x^2 \neq y^2 \Rightarrow x \neq y$ and $x \neq -y$

$\boxed{D = \{(x,y) \mid x^2 \neq y^2\} \text{ or } \{(x,y) \mid x \neq \pm y\}}$

3) $f(x,y) = \dfrac{\ln(1+x)}{\sin y}$ Defined when $1 + x > 0$ and $y \neq n\pi$ $n = 0, \pm 1, \pm 2, \ldots$

$\boxed{D = \{(x,y) \mid x > -1 \text{ and } y \neq n\pi,\ n = 0, \pm 1, \pm 2, \ldots\}}$

4) $f(x,y) = \sqrt{x^2 + y^2}$ Defined when $x^2 + y^2 \geq 0$; this is always true. $\boxed{D = \mathbb{R}^2}$

<u>Ex</u> Determine the Range of each of the following functions

1) $f(x,y) = y/x$ For a given $x \neq 0$ we can make $f(x,y)$ take on any real value, including zero, so $\boxed{f \text{ is any real}}$

2) $f(x,y) = \sqrt{1-x^2-y^2}$ $\boxed{f \geq 0}$

3) $f(x,y) = \cos(xy)$ $\boxed{-1 \leq f \leq 1}$

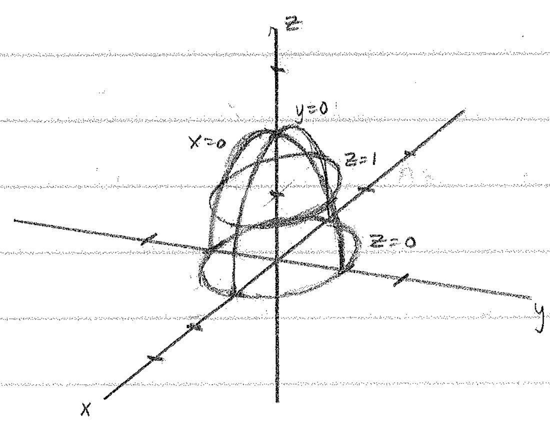
4) $f(x,y) = e^{x^2 y}$ Since x^2y can take on any real value we find $\boxed{0 < f < \infty}$

<u>Ex</u> Compute $f(3,-5)$ if $f(x,y) = x^2+y-2$

$$f(3,-5) = (3)^2+(-5)-2 = 9-5-2 = 2$$

Ex Sketch the indicated traces

1) $f(x,y) = 2\sqrt{1-x^2-y^2}$ $x=0,\ y=0,\ z=0,\ z=1$



$x=0 \rightarrow z = 2\sqrt{1-y^2} \rightarrow \frac{z^2}{4}+y^2=1$

$y=0 \rightarrow z = 2\sqrt{1-x^2} \rightarrow \frac{z^2}{4}+x^2=1$

$z=0 \rightarrow x^2+y^2=1$

$z=1 \rightarrow \frac{1}{4} = 1-x^2-y^2 \rightarrow x^2+y^2 = 3/4$

Functions of Several Variables

As you have observed, sketches of surface are difficult to draw and, while the give good qualitative insight into the shape and structure of the surface, they are not able to provide much useful quantitative information.

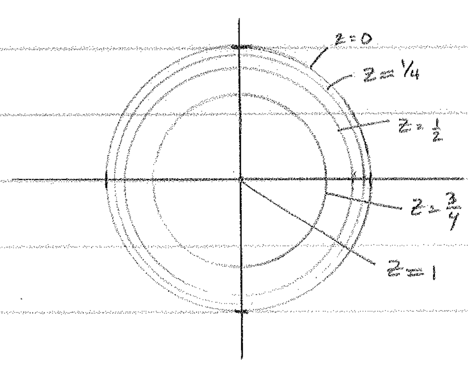
In the case of functions of two variables, contour plots are often used. Because these are 2D plots, they can be used for quantitative data.

Given a function $f(x,y)$, a contour plot is obtained by plotting multiple traces of $z = f(x,y)$ for different values of z.

When the spacing between these z values is constant the plot can convey useful information regarding the shape of the surface.

Ex Produce a contour plot of $f(x,y) = \sqrt{1-x^2-y^2}$

The domain of f is $\{(x,y) \mid x^2+y^2 \le 1\}$, i.e. the unit disk. The range of f is $0 \le f \le 1$. We'll plot traces for $z=0$, $z=1/4$, $z=1/2$, $z=3/4$, and $z=1$



$$z = f(x,y) = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$x^2+y^2 = 1-z^2 = \begin{cases} 1-0 = 1 = r^2 \to 1 \\ 1-\frac{1}{16} = \frac{15}{16} = r^2 \to \frac{\sqrt{15}}{4} \\ 1-\frac{1}{4} = \frac{3}{4} = r^2 \to \frac{\sqrt{3}}{2} \\ 1-\frac{9}{16} = \frac{7}{16} = r^2 \to \frac{\sqrt{7}}{4} \\ 1-1 = 0 = r^2 \to 0 \end{cases}$$

This is the contour plot of a hemisphere.
- the contour lines are close together when the surface is "steep"
- the contour lines are far apart when the surface is nearly "flat".

Examples
- topographic maps