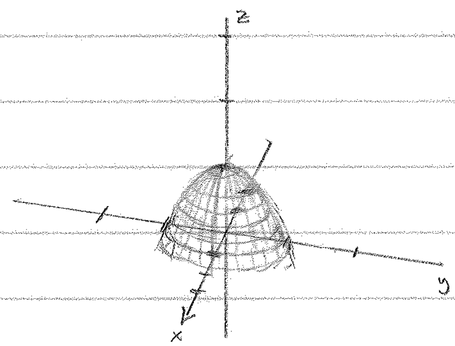
Partial Derivatives

Consider the paraboloid $z = f(x,y) = 1 - x^2 - y^2$



What is the "slope" of the surface at $(x,y) = (0,0)$?

What is the "slope" at $(x,y) = (1,0)$ or at $(x,y) = (0,1)$?

Over the next several classes we will explore ways to answer these questions.

We begin with the concept of <u>partial derivative</u>.

The <u>partial derivative of f with respect to x</u> is

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

and is defined for all (x,y) where this limit exists.

$\frac{\partial f}{\partial x}$ measures the rate of change of f in the x direction at the point (x,y).

Similarily, the partial derivative of f with respect to y is

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

We sometimes use the notation $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$

Partial Derivatives

Ex Let $f(x,y) = 1 - xy$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{(1-(x+h)y) - (1-xy)}{h}$$

$$= \lim_{h\to 0} \frac{1 - xy - hy - 1 + xy}{h} = \lim_{h\to 0} \frac{-hy}{h} = \lim_{h\to 0} -y = -y$$

So $\boxed{\frac{\partial f}{\partial x} = -y.}$ Notice that we treated y as if it was a constant since it is independent of x.

Compute $\frac{\partial f}{\partial x}$ by taking the derivative of f wrt x, treating y as a constant.

We can use this same reasoning to compute $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(1 - xy) = 0 - x = -x$$

Ex Find f_x and f_y if $f(x,y) = 1 - x^2 - y^2$

$$f_x = 0 - 2x - 0 = -2x, \quad f_y = 0 - 0 - 2y = -2y$$

From our motivating problem, $f_x(0,0) = f_y(0,0) = 0$ so f is not changing in either the x or y direction at $(0,0)$.

$$f_x(1,0) = -2,\ f_y(1,0) = 0 \quad \text{and} \quad f_x(0,1) = 0,\ f_y(0,1) = -2$$

Partial Derivatives

Ex $f(x,y) = x\sin y + \frac{x}{y}$

$$\frac{\partial f}{\partial x} = \sin y + \frac{1}{y}$$

$$\frac{\partial f}{\partial y} = x\cos y - \frac{x}{y^2}$$

Ex $f(x,y) = \sqrt{1+e^{xy^2}}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(1+e^{xy^2}\right)^{-1/2}\left(e^{xy^2}\cdot y^2\right) = \frac{y^2 e^{xy^2}}{2\sqrt{1+e^{xy^2}}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\left(1+e^{xy^2}\right)^{-1/2}\left(e^{xy^2}\cdot 2xy\right) = \frac{xy\, e^{xy^2}}{\sqrt{1+e^{xy^2}}}$$

Ex $f(x,y) = x\cos(xy^2+x^2y)$

$$f_x = \cos(xy^2+x^2y) + x\left(-\sin(xy^2+x^2y)\right)(y^2+2xy)$$
$$= \cos(xy^2+x^2y) - (xy^2+2x^2y)\sin(xy^2+x^2y)$$

$$f_y = -x\sin(xy^2+x^2y)(2xy+y) = -(2x^2y+xy)\sin(xy^2+x^2y)$$

Ex $f(x,y) = x\sin xy$

$$f_x = 1\cdot\sin xy + x\cos xy\cdot y = \sin xy + xy\cos xy$$
$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \cos xy\cdot y + y\cos xy + xy(-\sin xy)\cdot y = 2y\cos xy - xy^2\sin xy$$
$$f_y = x\cos xy\cdot x = x^2\cos xy$$
$$f_{yy} = -x^2\sin xy\cdot x = -x^3\sin xy$$

We can also "mix" derivatives

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = f_{xy} = \frac{\partial}{\partial y}(\sin xy + xy\cos xy) = \cos xy \cdot x + x\cos xy + xy(-\sin xy)\cdot x$$
$$= 2x\cos xy - x^2 y \sin xy$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = f_{yx} = \frac{\partial}{\partial x}(x^2\cos xy) = 2x\cos xy + x^2(-\sin xy)\cdot y$$
$$= 2x\cos xy - x^2 y \sin xy$$

Note $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are equal for most functions f

Thm If f_{xy} and f_{yx} are continuous on an open set containing (a,b) then $f_{xy}(a,b) = f_{yx}(a,b)$.