Tangent Planes and Linear Approximations

A typical first semester calculus problem is

Find the equation of the line tangent to
$y = 3 - x^2$ at the point $(1,2)$.

The slope of the tangent line is found by $y'(1) = -2x\big|_{x=1} = -2$
We can now use the point-slope line equation

$$y - y_1 = m(x - x_1)$$
$$y - 2 = (-2)(x - 1)$$
$$y = -2x + 4$$

We did this enough that we developed a general formula

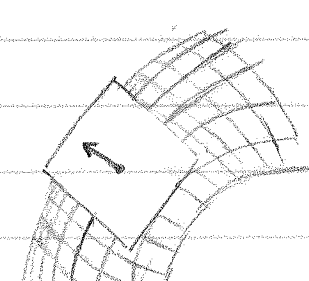
$$y = L(x) = f(a) + f'(a)(x - a)$$

called the <u>linear approximation</u> to $f(x)$ at $x = a$.

Now we are working with surfaces and we want to do something similar. Given a surface $z = f(x,y)$, we can find a plane tangent to the surface at $(x,y) = (a,b)$.

How can we find the equation of the plane? We need

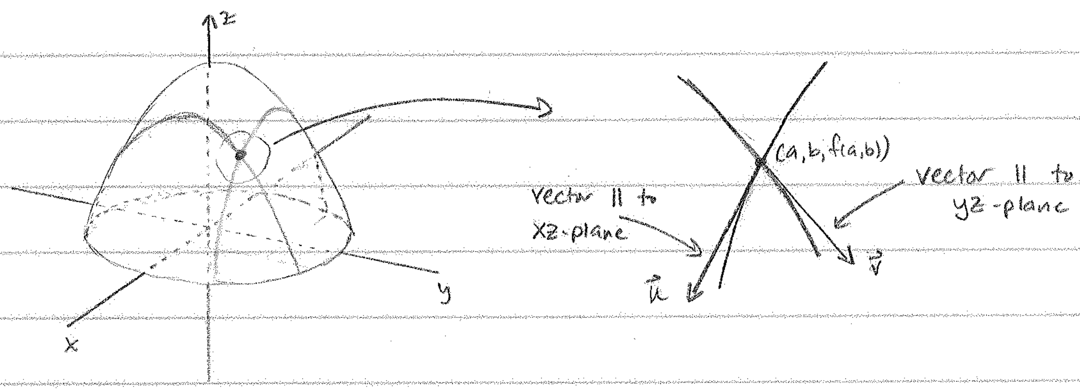
1. a point on the plane
2. a vector normal to the plane



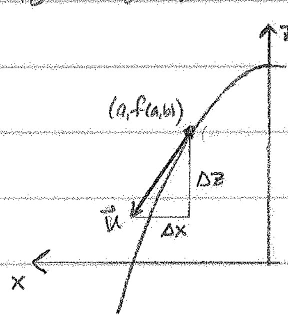
Tangent Planes and Linear Approximations

- the point of tangency is on both the surface and the plane

- How can we find a vector normal to the plane? If we know two vectors in the plane that point in different directions we can find their cross product to produce a normal vector.



To find $\vec{u}$:



view looking toward origin from +y axis

At point of tangency we want

$$\frac{\partial f}{\partial x}\bigg|_{(a,b)} = f_x(a,b) = \frac{\Delta z}{\Delta x}$$

$$\Delta x\,\hat{i} + \Delta z\,\hat{k} = \langle \Delta x, 0, \Delta z \rangle$$

we can scale this by $1/\Delta x$ to get

$$\vec{u} = \langle 1, 0, \frac{\Delta z}{\Delta x} \rangle = \langle 1, 0, f_x(a,b) \rangle$$

Since $\vec{u}$ is tangent to a trace of the surface, it is in the plane tangent to the surface.

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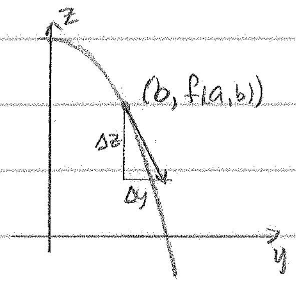
We can repeat to find $\vec{v}$.



$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b) = \frac{\Delta z}{\Delta y}$$

$$\Delta y\,\hat{j} + \Delta z\,\hat{k} = \langle 0, \Delta y, \Delta z\rangle$$

Scaling by $1/\Delta y$ we have

$$\vec{v} = \langle 0, 1, \tfrac{\Delta z}{\Delta y}\rangle = \langle 0, 1, f_y(a,b)\rangle$$

To find a normal vector to the plane we form the cross product. Either $\vec{u}\times\vec{v}$ or $\vec{v}\times\vec{u}$ will work, but the signs work out more nicely if we use $\vec{v}\times\vec{u}$.

$$\vec{n} = \vec{v}\times\vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & f_y(a,b) \\ 1 & 0 & f_x(a,b) \end{vmatrix} = f_x(a,b)\,\hat{\imath} - (-f_y(a,b))\hat{\jmath} + (-1)\hat{k}$$

$$= \langle f_x(a,b), f_y(a,b), -1\rangle$$

Using the equation of a plane $Ax + By + Cz + D = 0$ where $\langle A, B, C\rangle$ is a normal vector to the plane, we have

$$f_x(a,b)\,x + f_y(a,b)\,y - z + D = 0$$

At the point $(a, b, f(a,b))$ of tangency we have

$$f_x(a,b)\,a + f_y(a,b)\,b - f(a,b) + D = 0$$

Using this to replace D, we find

$$f_x(a,b)\,x + f_y(a,b)\,y - z - f_x(a,b)\,a - f_y(a,b)\,b + f(a,b) = 0$$

or

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Tangent Lines and Linear Approximations

Note: - This equation specifies a plane tangent to the surface $z = f(x,y)$ at the point $(x,y) = (a,b)$

- The tangent plane is a <u>linear approximation</u> to the function $f(x,y)$ near $(x,y) = (a,b)$

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- The <u>normal line</u> (parallel to the normal vector and through the point $(a, b, f(a,b))$) is given parametrically by

$$\boxed{\begin{aligned} x &= a + f_x(a,b)\,t \\ y &= b + f_y(a,b)\,t \\ z &= f(a,b) - t \end{aligned}}$$

Ex. Find the equations of the tangent plane and the normal line to the surface $z = x^2 - xy^2$ at the point $(2, 1, 2)$

$$f_x(2,1) = 2x - y^2 \Big|_{(2,1)} = 4 - 1 = 3$$
$$f_y(2,1) = 0 - 2xy \Big|_{(2,1)} = -4$$

$$\boxed{z = 2 + 3(x-2) - 4(y-1)}$$ Tangent Plane

$$\boxed{\begin{aligned} x &= 2 + 3t \\ y &= 1 - 4t \quad , \quad \frac{x-2}{3} = \frac{y-1}{-4} = \frac{z-2}{-1} \\ z &= 2 - t \end{aligned}}$$ Normal line