The Chain Rule

In some cases we have $z = f(x,y)$ but x and y are themselves functions of another variable, e.g. t. In this case

$$z = f(x(t), y(t)) = g(t)$$

and we may want to find how z changes with respect to t, i.e. find dz/dt.

$$\frac{dz}{dt} = \frac{d}{dt} g(t) = g'(t) = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$

if $\Delta z = f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))$.

From our work with linear approximations we can see

$$f(x,y) - f(a,b) \cong f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

or $f(x,y) - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + \varepsilon_1(x-a) + \varepsilon_2(y-b)$

↑ strict equality is possible because of ε_1 and ε_2 terms: $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(x,y) \to (a,b)$

If we replace x by $x(t+\Delta t)$, y by $y(t+\Delta t)$, a by $x(t)$, b by $y(t)$ and note $\Delta x = x - a = x(t+\Delta t) - x(t)$, $\Delta y = y - b = y(t+\Delta t) - y(t)$ we have

$$\Delta z = f_x(x(t), y(t))\,\Delta x + f_y(x(t), y(t))\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

or

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

So $$\frac{d}{dt} f(x(t), y(t)) = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

The Chain Rule

$$= \frac{\partial f}{\partial x}\left(\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}\right) + \frac{\partial f}{\partial y}\left(\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\right) + \lim_{\Delta t\to 0}\left(\varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}\right)$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \left(\lim_{\Delta t\to 0}\varepsilon_1\right)\frac{dx}{dt} + \left(\lim_{\Delta t\to 0}\varepsilon_2\right)\frac{dy}{dt}$$

Notice as $(x(t+\Delta t), y(t+\Delta t)) \to (x(t), y(t))$ we know $\varepsilon_1 \to 0$ & $\varepsilon_2 \to 0$

but this just means if $\Delta t \to 0$ then $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, so...

$$\boxed{\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$ The Chain Rule

Ex Suppose $f(x,y) = 2x - x^2y$ and $x(t) = t^2$, $y(t) = \sin t$

Find $\frac{df}{dt}$.

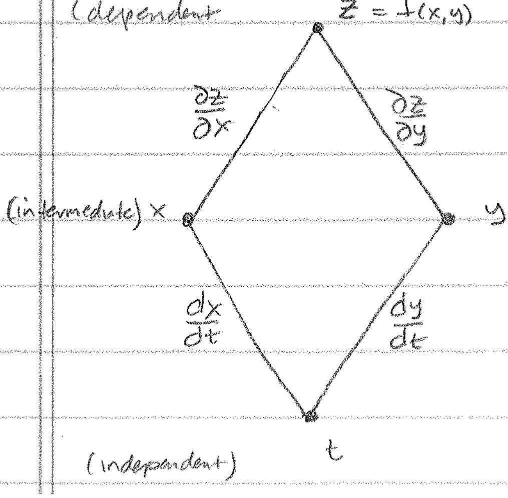
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{df}{dy} = (2-2xy)(2t) + (-x^2)(\cos t)$$

We have our answer but it is mixed in terms of variables so we'll express everything in terms of t.

$$\frac{df}{dt} = (2 - 2t^2\sin t)(2t) - (t^2)^2\cos t$$

$$\boxed{\frac{df}{dt} = 4t - 4t^3\sin t - t^4\cos t}$$

Sometimes a diagram is helpful...



$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}\frac{dx}{dt}}_{\text{path from } z \text{ to } t \text{ through } x} + \underbrace{\frac{\partial z}{\partial y}\frac{dy}{dt}}_{\text{path from } z \text{ to } t \text{ through } y}$$

The Chain Rule

The chain rule extends to functions of more than two variables in a natural way.

$$\frac{d}{dt} f(x,y,z) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

We also need the chain rule when there is more than independent variable.

$$z = f(x,y)$$

$$x = x(s,t)$$

$$y = y(s,t)$$

The chain Rule for this case is

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$

Ex. Polar Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

For any function $f(x,y)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = f_x(-r\sin\theta) + f_y\, r\cos\theta$$

$$= -f_x r\sin\theta + f_y r\cos\theta$$

The Chain Rule

Computing $\frac{\partial^2 f}{\partial r^2}$ needs some care...

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left[f_x \cos\theta + f_y \sin\theta\right]$$

$$= \frac{\partial f_x}{\partial r}\cos\theta + f_x \cancelto{0}{\frac{\partial \cos\theta}{\partial r}} + \frac{\partial f_y}{\partial r}\sin\theta + f_y \cancelto{0}{\frac{\partial \sin\theta}{\partial r}}$$

$$= \left[\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r}\right]\cos\theta + \left[\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r}\right]\sin\theta$$

$$= \left[f_{xx}\cos\theta + f_{xy}\sin\theta\right]\cos\theta + \left[f_{yx}\cos\theta + f_{yy}\sin\theta\right]\sin\theta$$

$$= f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta$$

$$= f_{xx}\cos^2\theta + f_{xy}\sin 2\theta + f_{yy}\sin^2\theta$$

Implicit Differentiation

Suppose we have $f(x,y,z)=0$ and we want to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. That is, we want to treat this as if z depends on x and y but that it is difficult to solve explicitly for z.

Let $w = f(x,y,z)$ and find $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y}$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

since $w=0 \to 0 = f_x \cdot 1 + f_y \cdot 0 + f_z \frac{\partial z}{\partial x}$ Since $\frac{\partial x}{\partial x} = 1$ and y and x are indep.

$$\therefore \boxed{\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}}$$

Similarly, we find $\boxed{\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}}$

The Chain Rule

Ex Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xz\cos y + e^{xyz^2} = 0$

Here $w = f(x,y,z) = xz\cos y + e^{xyz^2} = 0$

$$\frac{\partial f}{\partial x} = f_x = z\cos y + yz^2 e^{xyz^2}$$
$$\frac{\partial f}{\partial y} = f_y = -xz\sin y + xz^2 e^{xyz^2}$$
$$\frac{\partial f}{\partial z} = f_z = x\cos y + 2xyz\, e^{xyz^2}$$

So
$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{z\cos y + yz^2 e^{xyz^2}}{x\cos y + 2xyz\, e^{xyz^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{\partial z} = \frac{xz\sin y - xz^2 e^{xyz^2}}{x\cos y + 2xyz\, e^{xyz^2}}$$