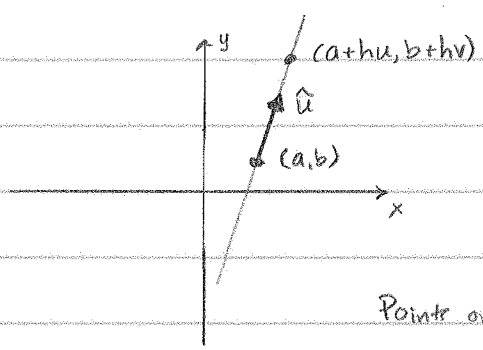
The Gradient and Directional Derivatives

Suppose we have $f(x,y) = 4 - x^2 - y^2$

We can compute the rate of change of f in the x and y directions at any point (a,b) with $f_x(a,b)$ and $f_y(a,b)$

How can we find the rate of change of f in other directions?

Let $\hat{u} = \langle u, v \rangle$ be a unit vector in the desired direction.



Note that if $\vec{u}$ is not a unit vector we can convert it to one with $\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$.

Points on the line through (a,b) and $\parallel$ to $\hat{u}$ are given by $(a+hu, b+hv)$ for some scalar h.

We basically want to measure the slope of the surface in the direction given by $\hat{u}$. The difference quotient will be

$$\frac{f(a+hu, b+hv) - f(a,b)}{\sqrt{(a+hu-a)^2 + (b+hv-b)^2}} = \frac{f(a+hu, b+hv) - f(a,b)}{h\sqrt{u^2+v^2}}$$

Where $h > 0$ and $\sqrt{u^2+v^2} = 1$ since $\hat{u}$ is a unit vector.

This gives

$$\frac{f(a+hu, b+hv) - f(a,b)}{h}$$

This becomes a derivative if we let $h \to 0$

The Gradient and Directional Derivatives

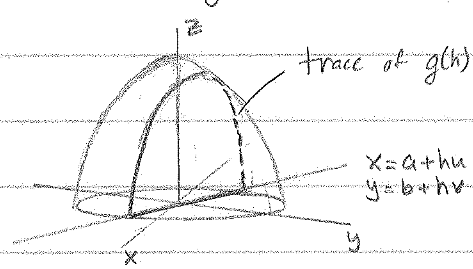
Def The directional derivative of $f(x,y)$ at the point (a,b) and in the direction of the unit vector $\hat{u} = \langle u, v \rangle$ is given by

$$D_u f(a,b) = \lim_{h \to 0} \frac{f(a+hu, b+hv) - f(a,b)}{h}$$

provided this limit exists.

As you probably remember from Calculus I, using a definition like this to compute derivatives can be cumbersome. We can use the chain rule to find a more useful form.

Let $g(h) = f(a+hu, b+hv)$. This defines a single variable function in the desired direction.



The line through (a,b) $\parallel$ to $\hat{u}$ is given parametrically by

$$x = a + hu, \quad y = b + hv.$$

We want to find $g'(0)$.

$$g'(h) = \frac{d}{dh} f(x,y) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v$$

$$g'(0) = f_x(a,b)\, u + f_y(a,b)\, v$$

$\therefore$ Thm: If f is differentiable at (a,b) and $\hat{u} = \langle u, v \rangle$ is a unit vector then

$$D_u f(a,b) = f_x(a,b)\, u + f_y(a,b)\, v$$

Ex $f(x,y) = 4 - x^2 - y^2$. Find $D_u f(1,1)$ in the direction $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$f_x = -2x, \quad f_y = -2y$$

$$D_u f(1,1) = (-2)\frac{1}{\sqrt{2}} + (-2)\frac{1}{\sqrt{2}} = -4/\sqrt{2} = \boxed{-2\sqrt{2}}$$

Ex Repeat, but use $\vec{u} = \langle 1, 3 \rangle$

$$\hat{u} = \frac{1}{\sqrt{1^2+3^2}} \langle 1, 3 \rangle = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

$$D_u f(1,1) = \frac{-2}{\sqrt{10}} + \frac{-6}{\sqrt{10}} = \boxed{\frac{-8}{\sqrt{10}}}$$

Def The gradient of $f(x,y)$ is the vector-valued function

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

provided the partial derivatives exist.

We read ∇f as "del f" and sometimes written "grad f"

The gradient allow us to express the directional derivative even more concisely.

$$\boxed{D_u f(x,y) = \nabla f(x,y) \cdot \hat{u}}$$

Ex. Let $f(x,y) = x^2 + xy$.

$$\nabla f(x,y) = (2x+y)\hat{\imath} + x\hat{\jmath}$$

At $(0,0)$ $\nabla f(0,0) = 0\hat{\imath} + 0\hat{\jmath} = \vec{0}$

At $(1,0)$ $\nabla f(1,0) = \hat{\imath} + \hat{\jmath}$

At $(0,1)$ $\nabla f(0,1) = \hat{\imath}$

Find all (x,y) for which $\nabla f(x,y) = \vec{0}$.

$$\begin{matrix} 2x+y=0 \\ x=0 \end{matrix} \implies x=0,\ y=0 \quad \text{only point}$$

Let $\vec{u} = \langle 2,1 \rangle$ Then $\hat{u} = \frac{1}{\sqrt{2^2+1^2}}\langle 2,1 \rangle = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$

$$D_u f(1,1) = \nabla f(1,1) \cdot \hat{u} = (3\hat{\imath} + \hat{\jmath}) \cdot \left(\tfrac{2}{\sqrt{5}}\hat{\imath} + \tfrac{1}{\sqrt{5}}\hat{\jmath}\right)$$
$$= \frac{6}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{7}{\sqrt{5}}$$

Recall that $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

$$D_u f(x,y) = \nabla f(x,y) \cdot \hat{u} = \|\nabla f(x,y)\| \|\hat{u}\| \cos\theta$$
$$= \|\nabla f(x,y)\| \cos\theta \quad \text{since } \|\hat{u}\| = 1$$

For a particular point $(x,y) = (a,b)$ we find that

① $D_u f(a,b) = 0$ when $\nabla f(a,b) = \vec{0}$ or $\theta = \pm \pi/2$

② The maximum of $D_u f(a,b)$ occurs when $\theta = 0$

③ The minimum of $D_u f(a,b)$ occurs when $\theta = \pi$

Q: What does it mean for $D_u f(x,y) = 0$? Describe this geometrically in terms of the surface $z = f(x,y)$.

Thm: Suppose that $f(x,y)$ is differentiable at (a,b). Then

① The maximum rate of change of f at (a,b) is $\|\nabla f(a,b)\|$, occurring in the direction of the gradient.

② The minimum rate of change of f at (a,b) is $-\|\nabla f(a,b)\|$, occuring in the direction opposite the gradient.

③ The rate of change of f at (a,b) is 0 in directions orthogonal to the gradient.

④ The gradient $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = c$ at every point (a,b) where $c = f(a,b)$.

Ex Find the directions of maximum and minimum rate of change of $f(x,y) = x\cos(xy)$ at $(1, \pi/2)$

$$\nabla f(x,y) = [\cos(xy) - xy\sin(xy)]\,\hat{\imath} - x^2\sin(xy)\,\hat{\jmath}$$

$$\nabla f(1,\pi) = [\cos\tfrac{\pi}{2} - \tfrac{\pi}{2}\sin\tfrac{\pi}{2}]\,\hat{\imath} - 1^2\sin\tfrac{\pi}{2}\,\hat{\jmath}$$

$$= (0 - \tfrac{\pi}{2})\hat{\imath} - \hat{\jmath} = -\tfrac{\pi}{2}\hat{\imath} - \hat{\jmath}$$

$$\|\nabla f(1,\pi)\| = \sqrt{(-\tfrac{\pi}{2})^2 + (-1)^2} = \sqrt{\tfrac{\pi^2}{4} + 1} = \frac{\sqrt{\pi^2+4}}{2}$$

Direction of max rate of change: $\langle -\tfrac{\pi}{2}, -1 \rangle$, Value $\frac{\sqrt{\pi^2+4}}{2}$

" min " $\langle \tfrac{\pi}{2}, 1 \rangle$, Value $-\frac{\sqrt{\pi^2+4}}{2}$

The definitions of directional derivative & gradient extend in the obvious way to functions of more than two variables.

Ex $f(x,y,z) = xz^2 + 2yz$

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$
$$= z^2\hat{\imath} + 2z\hat{\jmath} + (2xz + 2y)\hat{k}$$

If $\hat{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ then

$$D_u f(x,y,z) = \frac{1}{\sqrt{3}}\left(z^2 + 2z + 2xz + 2y\right)$$

Ex Find the equation of the tangent plane and normal line to the surface $z = x^2 + xy$ at the point $(1,2,3)$.

Let $f(x,y,z) = x^2 + xy - z = 0$. This is a "level surface" much the same as a level curve is for a function of two variables. Just as the 2D gradient points in the direction of greatest change; orthogonal to the level curve, the 3D gradient points in the direction of greatest change, orthogonal to the level surface.

$\Rightarrow$ $\nabla f(a,b,c)$ is normal to the surface at $(a,b,$

$$\nabla f(x,y,z) = (2x+y)\hat{i} + x\hat{j} - 1\hat{k}$$

$$\nabla f(1,2,3) = (2+2)\hat{i} + 1\hat{j} - 1k = 4\hat{i} + \hat{j} - \hat{k}$$

$$4(x-1) + 1(y-2) - 1(z-3) = 0$$

$$4x - 4 + y - 2 - z + 3 = 0$$

$$\boxed{4x + y - z - 3 = 0}$$

Summary

	Directional Derivative	Gradient
2D form	$D_u f(x,y)$	$\nabla f(x,y)$
3D form	$D_u f(x,y,z)$	$\nabla f(x,y,z)$
Def.	$D_u f(x,y) = \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v$ if $\hat{u} = \langle u, v \rangle$ unit vector $= \nabla f(x,y) \cdot \hat{u}$	$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$
Type	Scalar	Vector
Meaning	rate of change in direction $\hat{u}$	direction & magnitude of greatest change