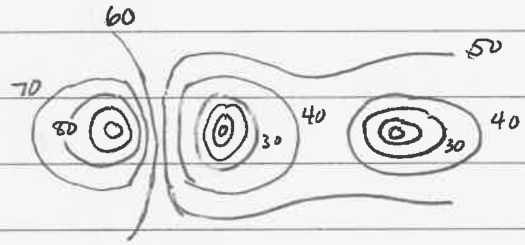
Extrema of Functions of Several Variables

Consider a surface with the contour plot:



Def: We call $f(a,b)$ a <u>local maximum</u> of f if there is an open disk R centered at (a,b), for which $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$. Similarly, $f(a,b)$ is a <u>local minimum</u> of f if there is a open disk R centered at (a,b), for which $f(a,b) \leq f(x,y)$ for all $(x,y) \in R$. In either case, $f(a,b)$ is a <u>local extremum</u> of f.

How can we find these extrema?

– at a local extremum, we expect $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

Def: The point (a,b) is a <u>critical point</u> of $f(x,y)$ if (a,b) is in the domain of f and either $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ or one or both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not exist at (a,b).

Ex $f(x,y) = xe^{-x^2-y^2}$

$$\frac{\partial f}{\partial x} = e^{-x^2-y^2} - 2x^2e^{-x^2-y^2} = (1-2x^2)e^{-x^2-y^2}$$

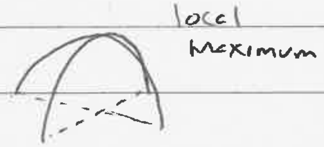
$$\frac{\partial f}{\partial y} = -2xye^{-x^2-y^2}$$

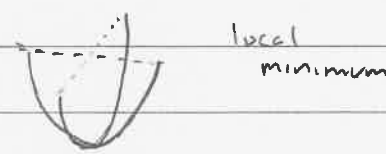
These both exist for all x & y, but $\frac{\partial f}{\partial x} = 0$ when $1-2x^2=0$ and $\frac{\partial f}{\partial y} = 0$ when $2xy = 0$. Therefore the critical points are $(-\frac{1}{\sqrt{2}}, 0)$ and $(\frac{1}{\sqrt{2}}, 0)$.

Extrema of Functions of Several Variables

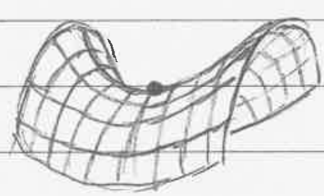
☆ Not every critical point corresponds to an extreme value.

One way to determine if a critical point locates an extremum is to graph the function and inspect the critical points. A local maximum will appear as a "hill" while a local minimum will appear as a "dale" (valley). Consider the traces





in both of these $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ have the same sign at the critical point. What happens if they have different signs?



<u>Def</u> The point $P(a,b,f(a,b))$ is a <u>saddle point</u> of $z=f(x,y)$ if (a,b) is a critical point of f and if every open disk centered at (a,b) contains points (x,y) in the domain of f for which $f(x,y) < f(a,b)$ and points (x,y) for which $f(x,y) > f(a,b)$.

Thm (Second Derivative Test)

Suppose $f(x,y)$ has continuous second-order partial derivatives in some open disk containing the point (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Define the discriminant D at (a,b) by

$$D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

i) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b))

ii) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b)

iii) If $D(a,b) < 0$, then f has a saddle point at (a,b)

iv) If $D(a,b) = 0$, then no conclusion can be drawn.

Ex $f(x,y) = xe^{-x^2-y^2}$

$$f_x = (1-2x^2)e^{-x^2-y^2} \qquad f_y = -2xy\, e^{-x^2-y^2}$$

$$f_{xx} = -4xe^{-x^2-y^2} - 2x(1-2x^2)e^{-x^2-y^2} = (4x^3-6x)e^{-x^2-y^2}$$

$$f_{yy} = -2xe^{-x^2-y^2} + 4xy^2e^{-x^2-y^2} = (4xy^2-2x)\, e^{-x^2-y^2}$$

$$f_{xy} = -2y(1-2x^2)e^{-x^2-y^2} = (4x^2y - 2y)\, e^{-x^2-y^2}$$

So $D = (4x^3-6x)(4xy^2-2x)e^{-2x^2-2y^2} - (4x^2y-2y)^2 e^{-2x^2-2y^2}$

$$D(\tfrac{1}{\sqrt{2}}, 0) = \left[\left(4\cdot\left(\tfrac{1}{\sqrt{2}}\right)^3 - 6\left(\tfrac{1}{\sqrt{2}}\right)\right)\left(-2\left(\tfrac{1}{\sqrt{2}}\right)\right) - 0\right]e^{-1} =$$

$$= \left(\frac{4}{2\sqrt{2}} - \frac{6}{\sqrt{2}}\right)(-\sqrt{2})\,1/e = 4/e > 0$$

$$f_{xx}(\tfrac{1}{\sqrt{2}}, 0) = \left(4\left(\tfrac{1}{\sqrt{2}}\right)^3 - 6\left(\tfrac{1}{\sqrt{2}}\right)\right)e^{-1/2} = \left(\frac{4}{2\sqrt{2}} - \frac{6}{\sqrt{2}}\right)/e^{1/2} = -\frac{4}{\sqrt{2}e^{1/2}} < 0$$

$\therefore$ $(\frac{1}{\sqrt{2}}, 0)$ is the location of a local maximum.

Now $D(-\frac{1}{\sqrt{2}}, 0) = \left[\left(4\left(-\tfrac{1}{\sqrt{2}}\right)^3 - 6\left(-\tfrac{1}{\sqrt{2}}\right)\right)\left(-2\left(-\tfrac{1}{\sqrt{2}}\right)\right) - 0\right]e^{-1}$

$$= \left(-\frac{4}{2\sqrt{2}} + \frac{6}{\sqrt{2}}\right)(\sqrt{2})\,1/e = 4/e > 0$$

$$f_{xx}(-\tfrac{1}{\sqrt{2}}, 0) = \left(4\left(-\tfrac{1}{\sqrt{2}}\right)^3 - 6\left(-\tfrac{1}{\sqrt{2}}\right)\right)e^{-1/2} = \left(-\frac{4}{2\sqrt{2}} + \frac{6}{\sqrt{2}}\right)/e^{1/2}$$
$$= \frac{4}{\sqrt{2}e^{1/2}} > 0$$

$\therefore$ $(-\frac{1}{\sqrt{2}}, 0)$ is the location of a local minimum.

Def: The absolute maximum of f on the region R is $f(a,b)$ if $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$; absolute minimum is defined similarly.

Extreme Value Thm: Suppose $f(x,y)$ is defined on the closed bounded region $R \subset \mathbb{R}^2$. Then f has both an abs. max. and an abs. min. on R. The absolute extremum may only occur at a critical point in R or on the boundary of R.

Ex. Find the absolute extrema of $f(x,y) = x^2 + 2y^2 - xy$ on the region bounded by $x=0$, $y=0$, $y = 1-x$.

We begin by identifying and classifying the critical points.

① $f_x = 2x - y = 0$ $\quad f_y = 4y - x = 0$

$y = 2x \longrightarrow 4(2x) - x = 0 \Rightarrow 7x = 0$ so $x = 0$, $y = 0$

② $f_{xx} = 2$, $f_{yy} = 4$, $f_{xy} = -1$

so $D(0,0) = 2 \cdot 4 - (-1)^2 = 8 - 1 = 7$

Since $D(0,0) > 0$ and $f_{xx}(0,0) = 2 > 0$,

$\boxed{f \text{ has a local minimum at } (0,0),\ f(0,0) = 0.}$

We now need to look for extreme values on the boundary.

On $x=0$: $f(0,y) = 2y^2 \Rightarrow g(y) = 2y^2$, $g'(y) = 4y = 0 \rightarrow$ $\boxed{y=0 \text{ crit pt. min}}$

$g''(y) = 4 > 0$

on $y=0$ $f(x,0) = x^2 \Rightarrow h(x) = x^2$, $h'(x) = 2x = 0 \rightarrow$ $\boxed{x=0 \text{ min.}}$

$h''(x) = 2 > 0$

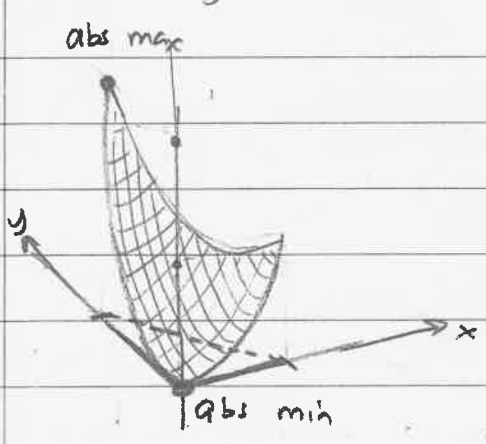
on $y = 1-x$ $f(x,1-x) = x^2 + 2(1-x)^2 - x(1-x)$

$= x^2 + 2 - 4x + 2x^2 - x + x^2$

$= 4x^2 - 5x + 2 \Rightarrow k(x) = 4x^2 - 5x + 2$

$k'(x) = 8x - 5 = 0$ $\boxed{x = 5/8 \text{ min}}$

$k''(x) = 8 > 0$



$\boxed{\begin{array}{l}\text{abs min at } f(0,0) = 0 \\ \text{abs max at } f(0,1) = 2\end{array}}$