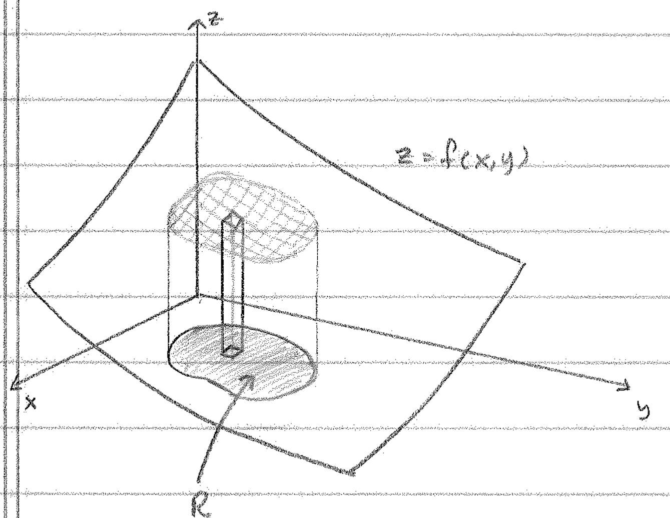
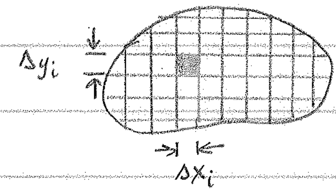
Double Integrals



Problem: Find the volume of the region above R and below $z = f(x,y)$.

Basic idea: split the volume into many small rectanglar prisms and sum the volume of these prisms.

We will partition R, a region of the xy-plane, into rectangles



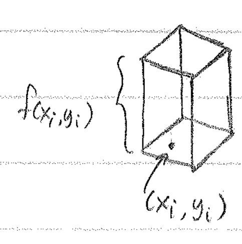
These small rectangles need not be square.

$$\Delta A_i = \Delta x_i \, \Delta y_i$$

If the partition is called P, we can define the norm of the partition $\|P\|$ to be the largest diagonal length of any rectangle in the partition. Then as $\|P\| \to 0$ the lengths of the sides of each rectangle must go to zero, and all $\Delta A_i \to 0$ as well.

To find the volume of a typical prism, we take (x_i, y_i) to be a point in the ith rectangle of P. Then $f(x_i, y_i)$ is the height of the prism.

$$\Delta V_i = f(x_i, y_i)\, \Delta A_i$$



Double Integrals

If we sum over all such prisms we will approximate the desired volume

$$V \approx \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

As $\|P\| \to 0$ we see $n \to \infty$, $\Delta x_i \to 0$, $\Delta y \to 0$ and

$$\sum_{i=1}^{n} \Delta A_i \to \text{Area of } R$$

Also $\displaystyle\sum_{i=1}^{n} \Delta V_i \to V$. This leads to the definition

<u>Def</u> For any function $f(x,y)$ defined on a bounded region $R \subseteq \mathbb{R}^2$ we define the <u>double integral</u> of f over R by

$$\iint_R f(x,y)\, dA = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

provided this limit exists. We say f is integrable over R.

Properties

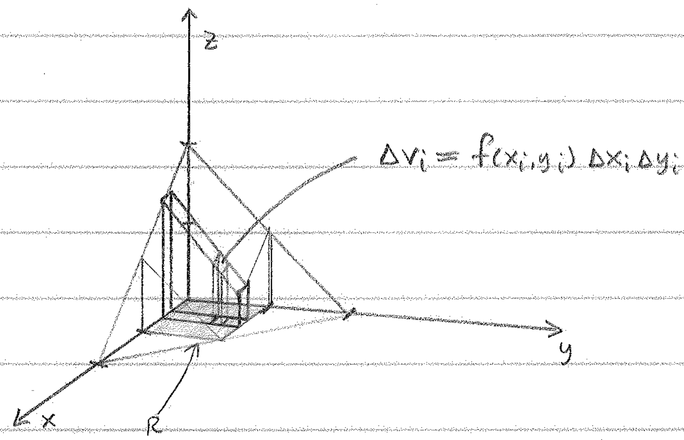
1. $\displaystyle\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$

2. $\displaystyle\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

3. $\displaystyle\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$ if $R = R_1 \cup R_2$ and $\emptyset = R_1 \cap R_2$

Double Integrals

Ex $f(x,y) = 2-x-y$ $\qquad R: 0 \le x \le 1,\ 0 \le y \le 1$



The volume of the slice is approximated by

$$V_{slice} \simeq \left[\sum_j f(x, y_j)\,\Delta y_j\right] \Delta x_i$$

where Δy_j are the lengths of Δy along the slice.

as $\Delta y_j \to 0$ we have

$$V_{slice} = \left[\int_0^1 (2-x-y)\,dy\right] \Delta x_i \qquad \text{where } x \text{ is fixed}$$

To find the total volume we "add" up all the slices

$$V \simeq \sum_i \left[\int_0^1 (2-x-y)\,dy\right] \Delta x_i$$

which, as $\Delta x_i \to 0$, we find

$$V = \int_0^1 \left[\int_0^1 (2-x-y)\,dy\right] dx$$

$$= \int_0^1 \left(2y - xy - \frac{y^2}{2}\right)\Big|_0^1 \, dx$$

$$= \int_0^1 \left[2 - x - \tfrac{1}{2} - 0 + 0 + 0\right] dx$$

$$= 2x - \frac{x^2}{2} - \frac{x}{2}\Big|_0^1 = 2 - \tfrac{1}{2} - \tfrac{1}{2} - 0 + 0 + 0 = \boxed{1}$$

We could, of course, sliced in the other direction, which would give

$$V = \int_0^1 \int_0^1 (2-x-y)\,dx\,dy$$

$$= \int_0^1 2x - \frac{x^2}{2} - yx \Big|_0^1 \,dy$$

$$= \int_0^1 2 - \frac{1}{2} - y \,dy$$

$$= 2y - \frac{y}{2} - \frac{y^2}{2} \Big|_0^1 = 2 - \frac{1}{2} - \frac{1}{2} = \boxed{1}$$

Thm (Fubini)

Suppose that f is integrable over the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$. Then

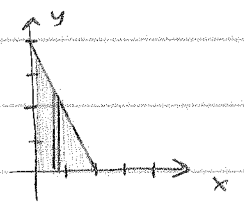
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

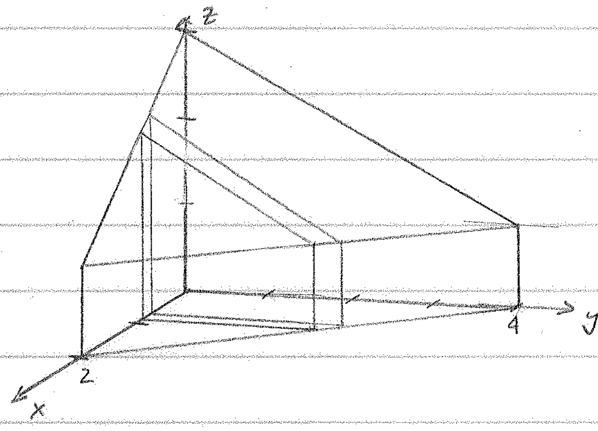
These are called "iterated integrals". Note how the limits of integration match the differentials.

What if R is not a rectangle?

Ex Evaluate $\iint_R (3 - x - \frac{1}{2}y)\,dA$ if R is the region bounded by $y=0$, $y = 4-2x$, $x=0$

① Draw R





The idea is the same as before, but now the length of the slice changes.

dV = volume of slice

$dV = (\text{Area of slice})(\text{thickness})$

$= \left(\int_0^{4-2x} (3 - x - \tfrac{1}{2}y)\, dy \right) dx$

$$V = \int dV = \int_0^2 \int_0^{4-2x} (3 - x - \tfrac{1}{2}y)\, dy\, dx$$

$$= \int_0^2 3y - xy - \frac{y^2}{4} \Big|_0^{4-2x} dx$$

$$= \int_0^2 \left[3(4-2x) - x(4-2x) - \frac{(4-2x)^2}{4} - 0 \right] dx$$

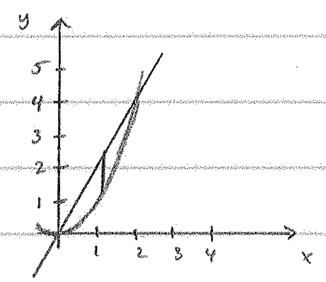
$$= \int_0^2 12 - 6x - 4x + 2x^2 - \frac{16 - 16x + 4x^2}{4} \Big] dx$$

$$= \int_0^2 (x^2 - 6x + 8)\, dx$$

$$= \frac{x^3}{3} - 3x^2 + 8x \Big|_0^2$$

$$= \frac{8}{3} - 12 + 16 - 0 = \frac{8}{3} + 4 = \frac{20}{3}$$

Ex Integrate $f(x,y) = x^2y + x$ over R if R is the region bounded by $y = 2x$ and $y = x^2$.



If we first integrate along y we find the lower and upper limits are $y = x^2$ and $y = 2x$

To find the limits on x, we want to integrate over the furthest x extent of R, in this case from $x=0$ to $x=2$

$$\iint_R x^2y + x \, dA = \int_0^2 \int_{x^2}^{2x} x^2y + x \, dy\, dx$$

$$= \int_0^2 x^2 \frac{y^2}{2} + xy \Big|_{x^2}^{2x} dx$$

$$= \int_0^2 2x^4 + 2x^2 - \frac{x^6}{2} - x^3 \, dx$$

$$= \frac{2}{5}x^5 + \frac{2}{3}x^3 - \frac{x^7}{14} - \frac{x^4}{4} \Big|_0^2$$

$$= \frac{64}{5} + \frac{16}{3} - \frac{128}{14} - 4 - [0]$$

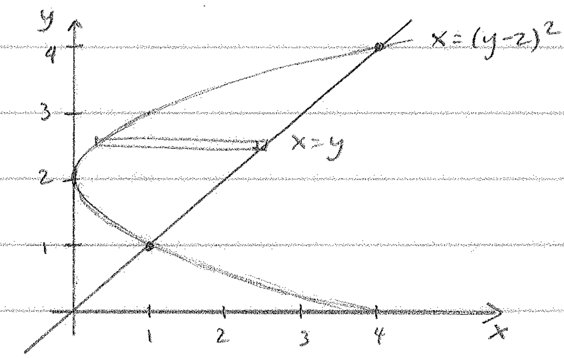
$$= \frac{64}{5} + \frac{16}{3} - \frac{64}{7} - 4 \qquad LCD = 105$$

$$= \frac{1344 + 560 - 960 - 420}{105}$$

$$= \boxed{\frac{524}{105}}$$

$$\approx 4.99$$

Ex Integrate $f(x,y) = 2y$ over R if R is the region bounded by $x = y$ and $x = (y-2)^2$



We will find it easier to slice along the x direction first.

limits on x: from $x=(y-2)^2$ on left to $x=y$ on right

limits on y: from $y=1$ to $y=4$ (points of intersection)

$$\iint_R 2y\,dA = \int_1^4\int_{(y-2)^2}^{y} 2y\,dx\,dy$$

$$= \int_1^4 2xy\Big|_{(y-2)^2}^{y} dy$$

$$= \int_1^4 2y^2 - 2(y^2-4y+4)y\,dy$$

$$= \int_1^4 2y^2 - 2y^3 + 8y^2 - 8y\,dy$$

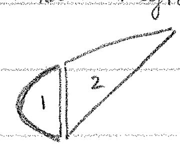
$$= \int_1^4 -2y^3 + 10y^2 - 8y\,dy$$

$$= -\tfrac{1}{2}y^4 + \tfrac{10}{3}y^3 - 4y^2\Big|_1^4$$

$$= -128 + \frac{640}{3} - 64 - \left[-\frac{1}{2} + \frac{10}{3} - 4\right]$$

$$= -\frac{375}{2} + 210 = \boxed{\frac{45}{2}}$$

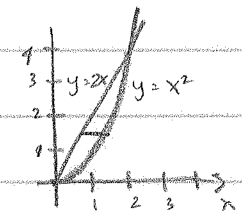
This would be difficult to integrate in the opposite order since we would need two regions because the lower bound function changes at $x=1$.



Double Integrals

Sometimes we are given an iterated integral and we want to change the order of integration.

<u>Ex</u> $\displaystyle\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$



① Draw R.

② label boundaries.

③ transpose slice

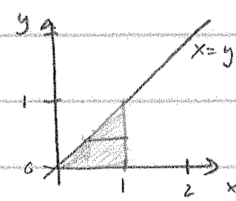
Integrate in x first → bounds are

$x = y/2$ to $x = \sqrt{y}$

Integrate in y second → bounds are 0 to 4

$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy.$$

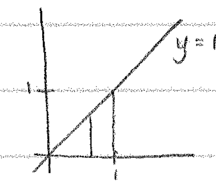
<u>Ex</u> $\displaystyle\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$



limits on x are y to 1

limits on y are 0 to 1

Unfortunately we don't know the antiderivative of e^{x^2}, so we cannot begin by finding it. Lets try reversing the order of integration to see if that helps...



limits on y : 0 to x

limits on x : 0 to 1

$$\int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 y e^{x^2}\Big|_0^x\,dx = \int_0^1 x e^{x^2}\,dx = \tfrac{1}{2} e^{x^2}\Big|_0^1 = \tfrac{1}{2}e^1 - \tfrac{1}{2}e^0$$

$$= \boxed{\tfrac{1}{2}(e-1)}$$