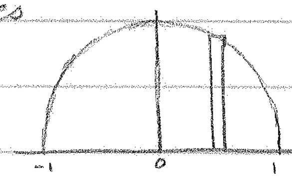
Double Integrals in Polar Coordinates

Ex $$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$



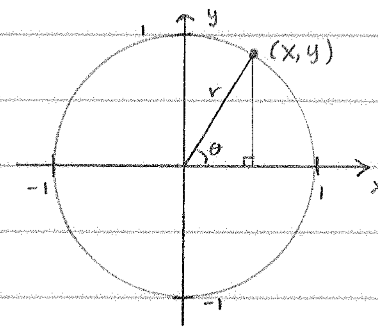
R is the semicircular region of radius 1 above the x axis

$$\int_{-1}^{1} x^2y + \frac{y^3}{3}\Big|_0^{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{2}\,dx$$

Now what?

It turns out that switching to polar coordinates is very helpful here.

$$x = r\cos\theta \qquad y = r\sin\theta$$



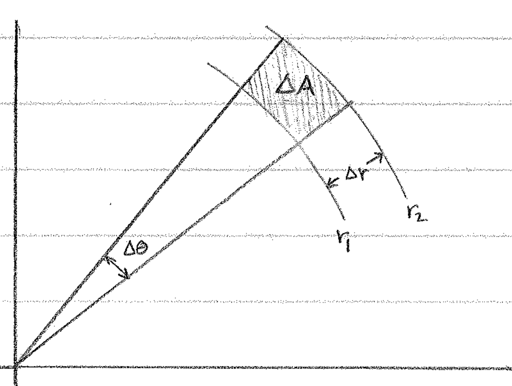
$$r^2 = x^2+y^2 \rightarrow r = \pm\sqrt{x^2+y^2}$$

General double integral in Cartesian coordinates

$$\iint_R f(x,y)\,dA$$

We need to

1) express dA in terms of r and θ
2) express $f(x,y)$ in terms of r and θ
3) express the limits of integration appropriately.



Area of sector  $= \frac{1}{2}\alpha r^2$

ΔA = Area outer sector — Area inner sector

$$= \frac{1}{2}\Delta\theta\, r_2^2 - \frac{1}{2}\Delta\theta\, r_1^2$$

$$= \frac{1}{2}(r_2^2 - r_1^2)\,\Delta\theta$$

$$= \frac{1}{2}(r_2+r_1)(r_2-r_1)\,\Delta\theta = \frac{r_1+r_2}{2}\,\Delta r\,\Delta\theta$$

Double Integrals in Polar Coordinates

$\frac{r_1+r_2}{2}$ is the average of r_1 and r_2. As we let $\Delta r \to 0$ we find $r_1 \to r_2 \to r$

As $\|P\| \to 0$ (both Δr and $\Delta\theta \to 0$) we find

$$\frac{r_1+r_2}{2} \to r, \quad \Delta r \to dr, \quad \Delta\theta \to d\theta$$

So
$$\Delta A \to r\,dr\,d\theta$$

We also note $f(x,y) = f(r\cos\theta, r\sin\theta) = f(r,\theta)$.

So
$$\iint_R f(x,y)\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta$$

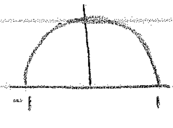
Thm (Fubini)

Suppose $f(r,\theta)$ is continuous on the region $R = \{(r,\theta) \mid \alpha \le \theta \le \beta\}$ and $g_1(\theta) \le r \le g_2(\theta)$, where $0 \le g_1(\theta) \le g_2(\theta)$ for all θ in $[\alpha,\beta]$.

Then
$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

$$\boxed{\text{WARNING: DON'T NEGLECT THE } r \text{ IN } dA = r\,dr\,d\theta}$$

Ex. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$. Note $x^2+y^2=r^2$

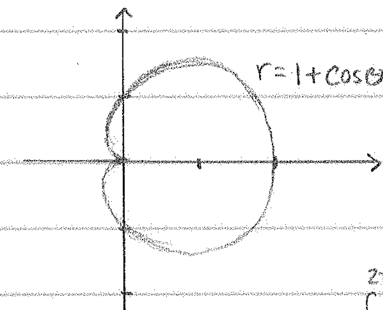


$$= \int_0^\pi \int_0^1 r^2\,r\,dr\,d\theta = \int_0^\pi\int_0^1 r^3\,dr\,d\theta = \int_0^\pi \frac{r^4}{4}\Big|_0^1\,d\theta = \int_0^\pi \frac{1}{4}\,d\theta$$

$$= \frac{1}{4}\theta\Big|_0^\pi = \boxed{\frac{\pi}{4}}$$

Double Integrals in Polar Coordinates

Ex Find the area inside the curve $r = 1+\cos\theta$



$$\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$$

$$\int_0^{2\pi} \frac{r^2}{2}\Big|_0^{1+\cos\theta} d\theta = \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2}\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \qquad \text{Since } \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

$$= \frac{\theta}{2} + \sin\theta + \frac{\theta}{4} + \frac{1}{8}\sin 2\theta \Big|_0^{2\pi}$$

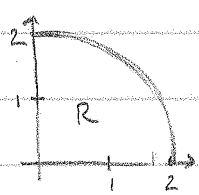
$$= \pi + 0 + \frac{\pi}{2} + 0 - (0+0+0+0)$$

$$= \boxed{\frac{3\pi}{2}}$$

When should you use polar coordinates in a double integral?

1) when there is circular symmetry
2) when we have expressions like x^2+y^2 that might be simpler when replaced by r^2
3) If cartesian integration seems to difficult

Ex $$\int_0^2\int_0^{\sqrt{4-x^2}} \frac{xy}{\sqrt{x^2+y^2}}\,dy\,dx \qquad x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2$$



$$\int_0^{\pi/2}\int_0^2 \frac{(r\cos\theta)(r\sin\theta)}{r}\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 r^2\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\cos\theta\sin\theta\Big|_0^2 d\theta = \frac{8}{3}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \qquad \begin{matrix} u=\sin\theta \\ du=\cos\theta\,d\theta \end{matrix}$$

$$= \frac{8}{3}\int_0^1 u\,du = \frac{8}{3}\frac{u^2}{2}\Big|_0^1 = \frac{8}{6} = \boxed{\frac{4}{3}}$$

Double Integrals in Polar Coordinates

Ex Find the volume of the solid whose base is given by $R: \{(x,y) \mid x^2+y^2 \le 1\}$ and whose upper boundary is the plane $5-x-y$.

$$\iint_R 5-x-y \, dA = \int_0^{2\pi}\int_0^1 (5-r\cos\theta-r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(5\cdot\frac{r^2}{2}-\frac{r^3}{3}\cos\theta-\frac{r^3}{3}\sin\theta\right)\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{5}{2}-\frac{1}{3}\cos\theta-\frac{1}{3}\sin\theta \; d\theta$$

$$= \frac{5}{2}\theta-\frac{1}{3}\sin\theta+\frac{1}{3}\cos\theta \Big|_0^{2\pi} = 5\pi-0+\frac{1}{3}-0+0-\frac{1}{3} = 5\pi$$