Triple Integrals

We can extend the ideas behind double integrals to include integrals along 3 axes (or even more).

As in the case of double integrals, the key to successful triple integration will be correctly specifying the region being integrated over.

Def For any function $f(x,y,z)$ defined on the bounded solid Q, we define the triple integral of f over Q by

$$\iiint_Q f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i$$

provided the limit exists and is the same for all choices of evaluation points in Q.

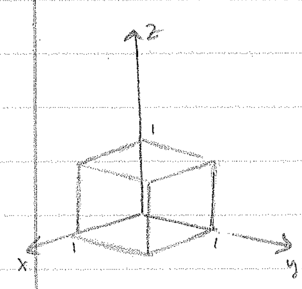
When Q is a rectangular box it is easy to construct a partition that fits the boundaries. In the general case, the partition is composed of rectangular solids contained strictly inside Q. The error made by doing this goes to zero as $\|P\| \to 0$.

Ex Integrate $f(x,y,z) = xyz$ over the unit cube.

$$\iiint_Q xyz\,dV = \int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx$$

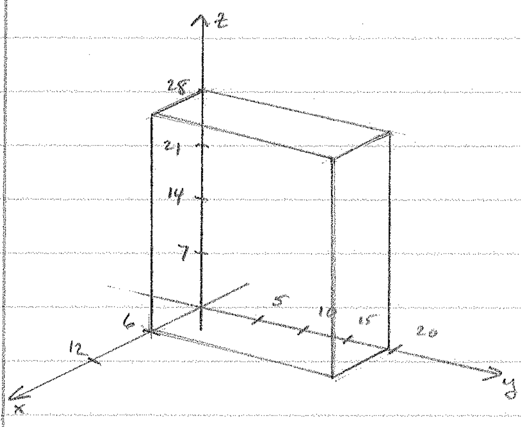
$$= \int_0^1\int_0^1 xy\frac{z^2}{2}\Big|_0^1\,dy\,dx = \int_0^1\int_0^1 \frac{xy}{2}\,dy\,dx$$

$$= \int_0^1 \frac{xy^2}{4}\Big|_0^1\,dx = \int_0^1 \frac{x}{4}\,dx = \frac{x^2}{8}\Big|_0^1 = \boxed{\frac{1}{8}}$$



Ex

Suppose the density of cereal in a 6 cm x 20 cm x 28 cm box is given by $\delta(x,y,z) = 1 - z/20 \ g/cm^3$. Find the total mass of cereal in the box.



$$\int_0^{28}\int_0^{20}\int_0^{6}(1 - z/20)\,dx\,dy\,dz$$

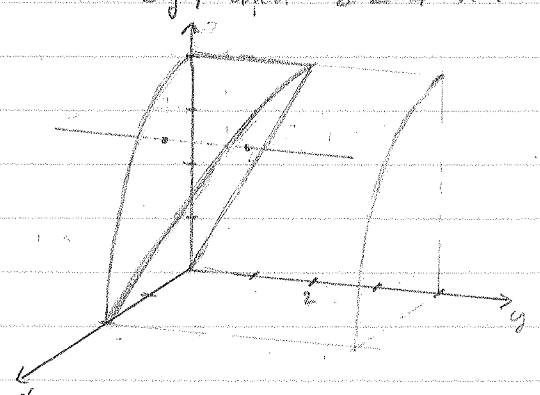
$$= \int_0^{28}\int_0^{20} x - \frac{xz}{20}\Big|_0^6 \,dy\,dz = \int_0^{28}\int_0^{20} 6 - \frac{3z}{10}\,dy\,dz$$

$$= \int_0^{28} 6y - \frac{3yz}{10}\Big|_0^{20} dz = \int_0^{28}(120 - 6z)\,dz$$

$$= 120z - 3z^2\Big|_0^{28} = 3360 - 2352 = 1008 \text{ g}$$

The mass of the cereal is 1008 g.

Ex Find the volume of the region bounded by $x=0$, $y=0$, $z=2y$, and $z=4-x^2$.



Integrate 1st along y from $y=0$ to $y = z/2$

Integrate next along z from $z=0$ to $z=4-x^2$

Integrate last along x from $x=0$ to $x=2$

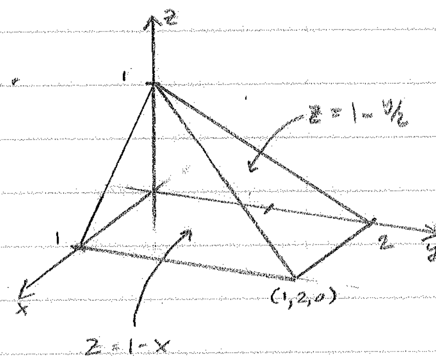
$$\int_0^2\int_0^{4-x^2}\int_0^{z/2} dy\,dz\,dx = \int_0^2\int_0^{4-x^2} \frac{z}{2}\,dz\,dx = \int_0^2 \frac{z^2}{4}\Big|_0^{4-x^2} dx$$

$$= \int_0^2 \frac{1}{4}(4-x^2)^2\,dx = \frac{1}{4}\int_0^2 16 - 8x^2 + x^4\,dx = \frac{1}{4}\left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^2$$

$$= \frac{1}{4}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \frac{1}{4}\left(\frac{480 - 320 + 96}{15}\right) = \frac{1}{4}\cdot\frac{256}{15} = \boxed{\frac{64}{15}}$$

Triple Integrals

<u>Ex</u> Find the volume of the solid shown.



First, get equations of planes that form the sides

$z = 0$ bottom

$y = 0$

$x = 0$

$z = 1 - y/2$ or $y + 2z = 2$

$z = 1 - x$ or $x + z = 1$

We will integrate in x or y first since the same formulas can be used over the entire cross sections perpendicular to x or y.

$$\int_0^1 \int_0^{2-2z} \int_0^{1-z} dx\, dy\, dz = \int_0^1 \int_0^{2-2z} (1-z)\, dy\, dz$$

$$= \int_0^1 (1-z)\, y \Big|_0^{2-2z} dz = \int_0^1 (1-z)(2-2z)\, dz = \int_0^1 2 - 4z + 2z^2\, dz$$

$$= 2z - 2z^2 + \tfrac{2}{3} z^3 \Big|_0^1 = 2 - 2 + \tfrac{2}{3} = \boxed{\tfrac{2}{3}}$$