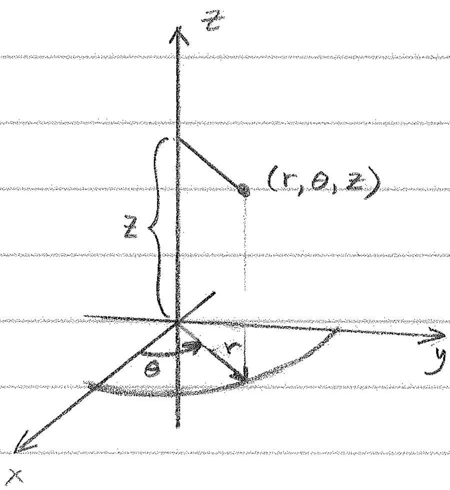
Cylindrical Coordinates

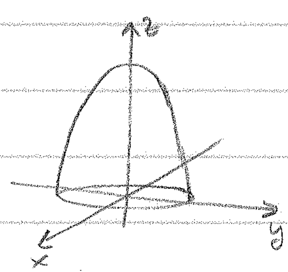
Just as in certain problems polar coordinates are more convenient than cartesian coordinates, cylindrical coordinates can be more advantageous than cartesian coordinates in three-dimensions.



Cylindrical coordinates are essentially polar coordinates in the xy-plane combined with the z-axis.

Ex Sketch the surface $z = 4 - r^2$

Notice that $r^2 = x^2 + y^2$ so this is $z = 4 - x^2 - y^2$



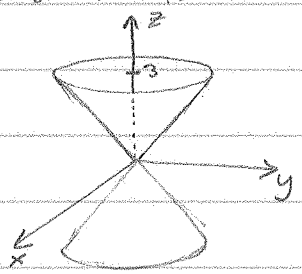
Ex Express $(x-1)^2 + (y+2)^2 = z$ in cylindrical coordinates

$$x^2 - 2x + 1 + y^2 + 4y + 4 = z$$

$$(x^2 + y^2) - 2x + 4y + 5 = z$$

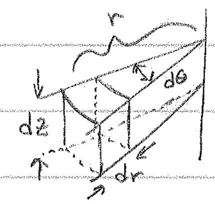
$$\boxed{r^2 - 2r\cos\theta + 4r\sin\theta + 5 = z}$$

Ex Find the volume in the region bounded by $z^2 = x^2 + y^2$ and $z = 3$.

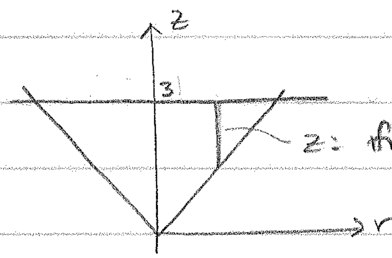


The region is an inverted cone.
In cylindrical coordinates the equation of the cone is $z^2 = r^2$ or just $z = r$.

Since we are integrating to find volume, we need an infinitesimal volume element dV



$$dV = r\,dz\,dr\,d\theta$$



z: from $z = r$ up to $z = 3$

$$V = \iiint_Q dV = \int_0^{2\pi}\int_0^3\int_r^3 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 rz\Big|_r^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 3r - r^2\,dr\,d\theta = \int_0^{2\pi} \frac{3r^2}{2} - \frac{r^3}{3}\Big|_0^3\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{27}{2} - 9\right)d\theta = \left(\frac{27}{2} - 9\right)\theta\Big|_0^{2\pi} = 2\pi\left(\frac{27-18}{2}\right)$$

$$= 9\pi$$

Check: $V_{cone} = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3}\pi(3)^2(3) = 9\pi$ ✓

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Ex Evaluate $\displaystyle\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{0}^{\sqrt[4]{x^2+y^2}} dz\,dy\,dx$

Starts off ok: $\displaystyle\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt[4]{x^2+y^2}\,dy\,dx$ but now what?

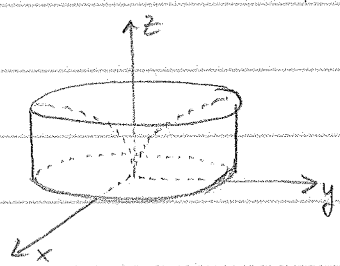
Sketching region Q

$z=0$ up to $z=\sqrt[4]{x^2+y^2} \Rightarrow z=\sqrt[4]{r^2}=\sqrt{r}$

$y=-\sqrt{16-x^2}$ to $y=\sqrt{16-x^2} \Rightarrow y^2=16-x^2 \Rightarrow x^2+y^2=16 \Rightarrow r=4$

$x=-4$ to 4 $\Rightarrow$ r goes from 0 to 4

θ goes from 0 to 2π



$dV = r\,dz\,dr\,d\theta$

z: goes from 0 up to $\sqrt{r}$

r: goes from 0 to 4

θ: goes from 0 to 2π

$$\int_0^{2\pi}\int_0^4\int_0^{\sqrt{r}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 rz\Big|_0^{\sqrt{r}}\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r^{3/2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{2}{5}r^{5/2}\Big|_0^4\,d\theta = \int_0^{2\pi} \frac{2}{5}\cdot 32\,d\theta$$

$$= \frac{64}{5}\cdot 2\pi = \frac{128\pi}{5}$$