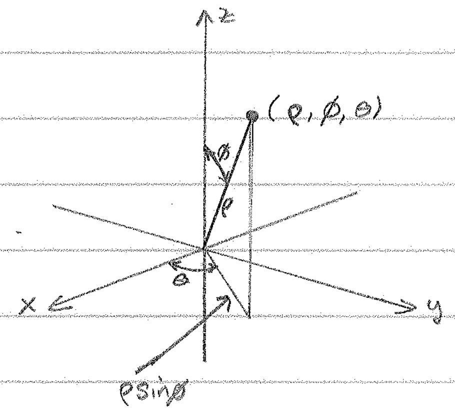
Spherical Coordinates

In some situations in $\mathbb{R}^3$ we have circular symmetry but this symmetry is with respect to the origin, not the z-axis. In this case the spherical coordinate system is used.



A point (x,y,z) can be described as (ρ, ϕ, θ) where

ρ – distance from origin to point.

ϕ – angle measured from the z axis to line from origin to point

θ – angle from x-axis to plane containing point and z-axis (measured counterclockwise)

We always take $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

Converting: $\rho^2 = x^2 + y^2 + z^2$

$\tan\theta = y/x$ (as in polar/cylindrical coordinates

$\rho\cos\phi = z$

$$\boxed{\rho = \sqrt{x^2+y^2+z^2}, \quad \phi = \arccos z/\rho, \quad \theta = \arctan y/x \; [\pm\pi]}$$

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi}$$

Spherical Coordinates

Ex Express $z = x^2+y^2$ in spherical coordinates

$$\rho\cos\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$$

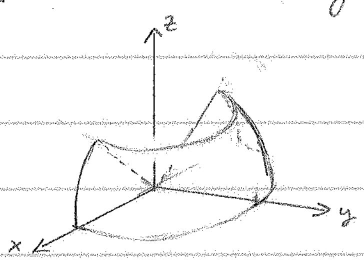
$$= \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)$$

$$\rho\cos\phi = \rho^2\sin^2\phi$$

$\rho=0$ is a solution, as is $\cos\phi = \rho\sin^2\phi$

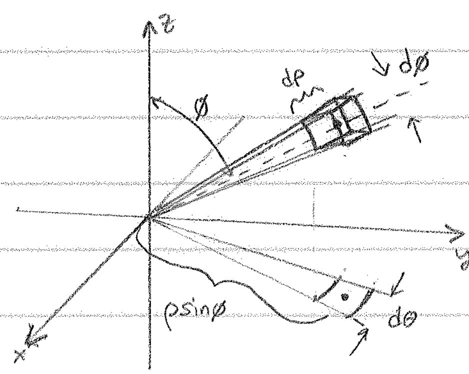
$$\boxed{\rho = \frac{\cos\phi}{\sin^2\phi}}$$

Ex Sketch the region $0\le\rho\le1$, $\frac{\pi}{4}\le\phi\le\frac{\pi}{2}$, $0\le\theta\le\pi$

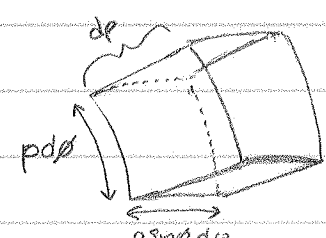


Ex Compute the volume of the region in the last example

Need the little piece of differential volume, dV



$$dV = (\rho\sin\phi\, d\theta)(\rho\, d\phi)\, d\rho$$



$$\boxed{dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

Now $V = \iiint_Q dV = \int_0^{\pi} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= \int_0^{\pi} \int_{\pi/4}^{\pi/2} \frac{\rho^3}{3} \sin\phi \Big|_{\rho=0}^{\rho=1} d\phi \, d\theta = \int_0^{\pi} \int_{\pi/4}^{\pi/2} \frac{\sin\phi}{3} \, d\phi \, d\theta$

$= \int_0^{\pi} -\frac{\cos\phi}{3} \Big|_{\pi/4}^{\pi/2} d\theta = \int_0^{\pi} \frac{-\cos\frac{\pi}{2} + \cos\frac{\pi}{4}}{3} d\theta = \int_0^{\pi} \frac{\sqrt{2}/2}{3} d\theta$

$= \frac{\sqrt{2}}{6} \theta \Big|_0^{\pi} = \boxed{\frac{\sqrt{2}\,\pi}{6}}$

Ex Integrate $f(x,y,z) = 3x$ over the region Q if Q is the portion of the sphere $\rho \le 4$ in the first octant

$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^4 3\rho \sin\phi \cos\theta \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= \int_0^{\pi/2} \int_0^{\pi/2} \frac{3\rho^4}{4} \Big|_0^4 \sin^2\phi \cos\theta \, d\phi \, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} 192 \sin^2\phi \cos\theta \, d\phi \, d\theta$

$= \int_0^{\pi/2} \int_0^{\pi/2} 192 \cdot \frac{1}{2}(1 - \cos 2\phi) \cos\theta \, d\phi \, d\theta = 96 \int_0^{\pi/2} \phi - \frac{1}{2}\sin 2\phi \Big|_0^{\pi/2} \cos\theta \, d\theta$

$= 96 \int_0^{\pi/2} (\frac{\pi}{2} - 0) \cos\theta \, d\theta = 96 \cdot \frac{\pi}{2} \cdot \sin\theta \Big|_0^{\pi/2} = \boxed{48\pi}$