Integration by Substitution

Find the following derivatives

$$\frac{d}{dx} e^{x^2} = e^{x^2} \cdot (2x) = 2x e^{x^2}$$

$$\frac{d}{dx} \cos(2x+1) = -\sin(2x+1) \cdot (2) = -2\sin(2x+1)$$

$$\frac{d}{dx} \ln(x^2+4) = \frac{1}{x^2+4} \cdot (2x) = \frac{2x}{x^2+4}$$

We used the <u>chain rule</u> to evaluate each of these derivatives.

Consider

$$\int 2x e^{x^2} dx$$

From the work above we know that this evaluates to $e^{x^2}+C$.
How could we do this without having just done the corresponding derivative?

Looking at the integrand we observe that $2x$ is the derivative of x^2.

We can try the substitution (change of variable) $u(x) = x^2$

$$u = x^2$$
$$du/dx = 2x$$

So

$$\int 2x e^{x^2} dx = \int \frac{du}{dx} e^u dx = \int e^u du = e^u + C$$
$$= e^{x^2} + C.$$

Ex $\int -2\sin(2x+1)\, dx$ Let $u = 2x+1$ so $du = 2\,dx$

$$= \int -\sin u\, du = \cos u + C = \underline{\cos(2x+1) + C}$$

Ex $\int \frac{2x}{x^2+4} dx$ Let $u = x^2+4$ so $du = 2x\,dx$

$$= \int \frac{1}{u} du = \ln|u| + C = \ln|x^2+4| + C = \underline{\ln(x^2+4) + C}$$

Integration by Substitution

Ex $\int x\sqrt{3x^2+1}\,dx$

Here we try $u = 3x^2+1$ so $du = 6x\,dx$

We don't have $6x$ in the integrand, but we can get it if we multiply by 6 inside the integral and divide by 6 outside.

$$\int x\sqrt{3x^2+1}\,dx = \frac{1}{6}\int 6x\sqrt{3x^2+1}\,dx = \frac{1}{6}\int \sqrt{u}\,du$$
$$= \frac{1}{6}\int u^{1/2}du = \frac{1}{6}\,\frac{2}{3}u^{3/2} + c$$
$$= \frac{1}{9}u^{3/2} + c = \underline{\frac{1}{9}(3x^2+1)^{3/2} + c}$$

Check $\frac{d}{dx}\,\frac{1}{9}(3x^2+1)^{3/2} + c = \frac{1}{9}\,\frac{3}{2}(3x^2+1)^{1/2}(6x)$
$$= \frac{1}{6}\sqrt{3x^2+1}\,(6x)$$
$$= x\sqrt{3x^2+1}$$

This substitution worked because we let u equal a quadratic function of x and there was a corresponding linear function of x in the integrand.

<u>Principle:</u> When choosing u, the derivative of u (up to a scalar multiple) must appear as a factor in the integrand.

Ex $\int \frac{x}{x^2+1}\,dx$ Let $u = x^2+1$ so $du = 2x\,dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$$
$$= \underline{\frac{1}{2}\ln|x^2+1| + c}$$

Integration by Substitution

Ex $\displaystyle\int \frac{1}{x^2+1}\,dx = \underline{\arctan x + C}$ Substitution is not used.

Ex $\displaystyle\int \frac{x^2}{x^2+1}\,dx$ $u = x^2+1 \Rightarrow du = 2x\,dx$

Cannot use this approach since we have x^2 in numerator rather than x.

We can divide
$$\begin{array}{r|l} & 1 \\ \hline x^2+1 & x^2 \\ & -(x^2+1) \\ \hline & -1 \end{array}$$
so

$$= \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \underline{x - \arctan x + C}$$

Ex $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ $u = \sqrt{x}$ $du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$

$$= 2\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx = 2\int \cos u\,du$$

$$= 2\sin u + C = \underline{2\sin\sqrt{x} + C}$$

Ex $\displaystyle\int x\sqrt{x-1}\,dx$ This one does not quite fit the rules shown so far — but we can use some algebraic manipulation

Let $u = x-1$. This means $du = dx$

What about the factor of x? If $u = x-1$ then $x = u+1$.

$$\int x\sqrt{x-1}\,dx = \int (u+1)\sqrt{u}\,du = \int u^{3/2} + u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$= \underline{\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}$$

Integration by Substitution

We can use substitution with definite integrals as well

Ex $\displaystyle\int_1^3 x\sqrt{x^2-1}\,dx \qquad u = x^2-1, \quad du = 2x\,dx$

We need to change the limits so that they correspond to the change in variable

$$x = 1 \rightarrow u = 1^2-1 = 0$$
$$x = 3 \rightarrow u = 3^2-1 = 8$$

So

$$\frac{1}{2}\int_1^3 2x\sqrt{x^2+1}\,dx = \frac{1}{2}\int_0^8 \sqrt{u}\,du = \frac{1}{2}\,\frac{2}{3}\,u^{3/2}\Big|_0^8 = \frac{1}{3}u^{3/2}\Big|_0^8$$
$$= \frac{1}{3}\left[8^{3/2} - 0^{3/2}\right] = \frac{1}{3}\left(\sqrt{8}\right)^3 = \underline{\frac{8}{3}\cdot(\sqrt{2})^3}$$

Ex $\displaystyle\int_0^{\pi/3} \tan x\,dx = \int_0^{\pi/3} \frac{\sin x}{\cos x}\,dx$

$$u = \cos x, \quad du = -\sin x\,dx$$
$$u(0) = 1, \quad u(\pi/3) = 1/2$$

$$= -\int_1^{1/2} \frac{1}{u}\,du = \int_{1/2}^1 \frac{1}{u}\,du = \ln|u|\Big|_{1/2}^1 = \ln 1 - \ln(1/2)$$
$$= 0 - (-\ln 2) = \underline{\ln 2}$$