Trigonometric Techniques of Integration

The relationships between the trigonometric functions often assist us in evaluating integrals involving them. In addition, they can also be helpful in evaluating some integrals that appear to have no relationship with them.

Ex $\int \sin^3 x \cos x \, dx$

Since $\cos x$ is the derivative of $\sin x$ we try a substitution $u = \sin x$, so $du = \cos x \, dx$

$$\int u^3 \, du = \frac{1}{4}u^4 + C = \boxed{\frac{1}{4}\sin^4 x + C}$$

Ex $\int \sin^2 x \cos^3 x \, dx$

We want to use the fact that sine and cosine are each other's derivative (well, up to a $\pm$ sign) but we don't have a single power of $\sin x$ or $\cos x$. But we can split $\cos^3 x$ up...

$$\int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x \,(1 - \sin^2 x) \cos x \, dx$$

$u = \sin x \quad du = \cos x \, dx$

$$= \int u^2(1-u^2)\, du = \int u^2 - u^4 \, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$\boxed{= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}$$

This same approach will work for $\int \sin^m x \cos^n x \, dx$ whenever m or n (or both) is an odd positive integer.

Why does one need to be odd?

- we need to pull one factor of $\sin x$ or $\cos x$ away from the rest of the integrand to be part of du
- we need the remaining part to have even power so it can be written in terms of $\sin^2 x$ or $\cos^2 x$.

Trigonometric Techniques of Integration

Ex $$\int \sin^5 x \cos^7 x \, dx = \int \sin^5 x \cos^6 x \cos x \, dx$$

$$= \int \sin^5 x (\cos^2 x)^3 \cos x \, dx$$

$$= \int \sin^5 x (1 - \sin^2 x)^3 \cos x \, dx \qquad u = \sin x, \; du = \cos x \, dx$$

$$= \int u^5 (1 - u^2)^3 du$$

$$= \int u^5 (1 - 3u^2 + 3u^4 - u^6) \, du = \int (u^5 - 3u^7 + 3u^9 - u^{11}) \, dx$$

$$= \frac{1}{6}u^6 - \frac{3}{8}u^8 + \frac{3}{10}u^{10} - \frac{1}{12}u^{12} + C$$

$$\boxed{= \frac{1}{6}\sin^6 x - \frac{3}{8}\sin^8 x + \frac{3}{10}\sin^{10} x - \frac{1}{12}\sin^{12} x + C}$$

What if both m and n are even?

We can use a trig identity to help.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Let $A = B$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

We can now use this, along with $\sin^2 A + \cos^2 A = 1$, to obtain

$$\cos^2 A = \cos 2A + \sin^2 A = \cos 2A + (1 - \cos^2 A)$$

$$2\cos^2 A = 1 + \cos 2A$$

$$\Rightarrow \cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

Similarly

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

Ex $$\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x \, dx$$

$$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin 2x + C}$$

Trigonometric Techniques of Integration

Ex $\int \sin^2 x \cos^2 x \, dx = \int \sin^2 x (1 - \sin^2 x) \, dx$

$= \int \sin^2 x \, dx - \int \sin^4 x \, dx$

Using $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ we have

$$= \int \tfrac{1}{2}(1 - \cos 2x) \, dx - \int \left[\tfrac{1}{2}(1 - \cos 2x)\right]^2 dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x - \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x - \tfrac{1}{4}\left[x - \sin 2x\right] - \tfrac{1}{4}\int \cos^2 2x \, dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{4}\int \tfrac{1}{2}(1 + \cos 4x) \, dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C = \boxed{\tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C}$$

Ex $\int \sec x \, dx$ This integral is easy — if you know the right trick! We merely need to multiply the integrand by 1 — but we need 1 in the right form...

$$\int \sec x \, dx = \int \sec x \overbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}^{1} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x) \, dx$

$$\int \frac{1}{u} \, du = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

Trigonometric Techniques of Integration

How about $\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$ Regular substitution won't work.

consider $\sqrt{1-x^2}$. For this to be real valued we know $x^2<1$ so $-1<x<1$. Let $x=\sin\theta$ with $-\pi/2<\theta<\pi/2$ [point out why we can't have $\theta=\pm\pi/2$].

$$x=\sin\theta, \quad dx=\cos\theta\, d\theta, \quad 1-x^2=1-\sin^2\theta=\cos^2\theta$$

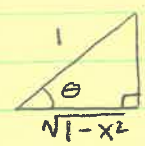
so the integral becomes

$$\int \frac{\cos\theta}{\sin^2\theta\sqrt{\cos^2\theta}}\,d\theta$$

from dx

$$= \int \csc^2\theta\, d\theta = -\cot\theta + C \qquad \text{[see table at start of 6.1]}$$

Now we need to convert back to x.



Since $\frac{x}{1}=\sin\theta$

$\cot\theta = \frac{\sqrt{1-x^2}}{x}$ so we have

$$\boxed{-\frac{\sqrt{1-x^2}}{x}+C}$$

We can use this approach for integrals involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ when $a>0$, at least sometimes.

If we have $\sqrt{a^2-x^2}$ we assume $x=a\sin\theta$ so

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2(1-\sin^2\theta)}=a\cos\theta$$

If we have $\sqrt{a^2+x^2}$ then we should try $x=a\tan\theta$, $-\pi/2<\theta<\pi/2$

$$\sqrt{a^2+x^2}=\sqrt{a^2+a^2\tan^2\theta}=\sqrt{a^2(1+\tan^2\theta)}=a\sec\theta$$

Since $1+\tan^2 x=\sec^2 x$ [divide $\cos^2 x+\sin^2 x=1$ by $\cos^2 x$]

If we have $\sqrt{x^2-a^2}$ then we should try $x=a\sec x$, with $0\le\theta<\pi/2$ or $\pi/2<\theta\le\pi$.

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2\theta-a^2}=\sqrt{a^2(\sec^2\theta-1)}=a\tan\theta$$

Trigonometric Techniques of Integration

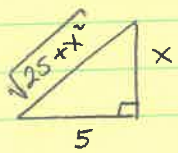
Ex $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,dx \qquad x = 5\tan\theta, \quad -\pi/2 < \theta < \pi/2$

$\qquad\qquad\qquad\qquad dx = 5\sec^2\theta\,d\theta$

$$= \int \frac{5\sec^2\theta}{\sqrt{25+25\tan^2\theta}}\,d\theta = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta =$$

$$= \ln|\sec\theta + \tan\theta| + C$$

Now, convert back to x



$\sec\theta = \dfrac{\sqrt{25+x^2}}{5}, \quad \tan\theta = \dfrac{x}{5}$ so

$$\boxed{\ln\left|\frac{\sqrt{25+x^2}+x}{5}\right| + C}$$

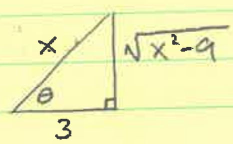
or $$\boxed{\ln\left|\sqrt{1+\left(\tfrac{x}{5}\right)^2} + \tfrac{x}{5}\right| + C}$$

Ex $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx \qquad x = 3\sec\theta \qquad$ for $0 \le x < \pi/2$ (need $x > 3$)

$\qquad\qquad\qquad\qquad dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{3\sec\theta\tan\theta}{\sqrt{9\sec^2\theta - 9}}\,dx = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$



$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

$$= \boxed{\ln\left|\frac{x}{3} + \sqrt{\left(\tfrac{x}{3}\right)^2 - 1}\right| + C}$$