Volume: Slicing, Disks, and Washers

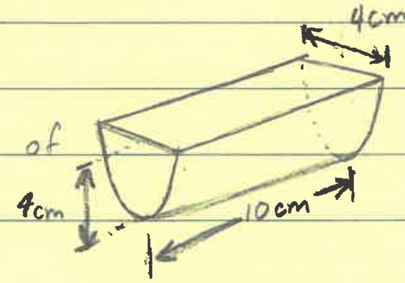
What volume formulas do you know?

Rectangular solid: $l \cdot h \cdot w$ cylinder: $\pi r^2 h$

Sphere: $\frac{4}{3}\pi r^3$ Cone: $\frac{1}{3}\pi r^2 h$

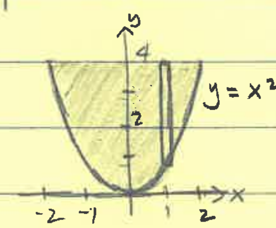
others?

How can we compute the volume of this solid?



Suppose we know the cross section is a parabola?

$$A = \int_{-2}^{2} (4 - x^2)\,dx = 4x - \frac{x^3}{3}\Big|_{-2}^{2}$$
$$= 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$$



So the cross sectional area is $\frac{32}{3}$ cm^2

The length of the solid perpendicular to the cross section is 10 cm, so the volume is $\frac{32}{3} \times 10 = \frac{320}{3}$ cm^3

≈ 106.667 cm^3

This idea is key to computing volumes of solids:

1. find the cross-sectional area of a "typical piece"
2. multiply by an infinitesimal thickness (often "dx" or "dy")
3. integrate along an axis perpendicular to the cross-section.

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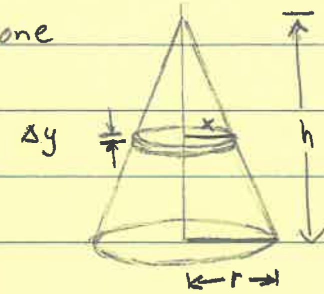
Ex: Find the volume of a right circular cone with height h and base radius r.

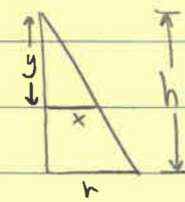


area of disk $= \pi x^2$

Volume of disk $= \pi x^2 \Delta y \rightarrow \pi x^2 dy$

if we consider dy to be an infinitesimal thickness

Since our cross section is perpendicular to the y-axis (and so dy is our thickness), we need to express our piece of volume in terms of y.



$\frac{x}{y} = \frac{r}{h}$ by similar triangles

$x = \frac{r}{h} y$

So

$$dV = \pi x^2 dy = \pi \left(\frac{r}{h} y\right)^2 dy = \frac{\pi r^2}{h^2} y^2 dy$$

To find volume, we integrate to "add up" all the little pieces of volume:

$$V = \int dV = \int_0^h \frac{\pi r^2}{h^2} y^2 \, dy$$

$$= \frac{\pi r^2}{h^2} \int_0^h y^2 \, dy = \frac{\pi r^2}{h^2} \frac{y^3}{3} \Big|_0^h$$

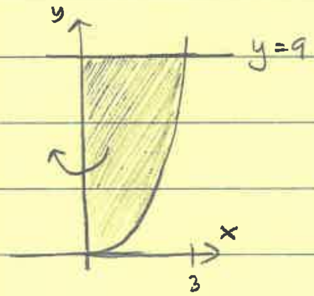
$$= \frac{\pi r^2}{h^2} \frac{h^3}{3} = \frac{\pi r^2}{3} h$$

$$= \frac{1}{3} \pi r^2 h \quad ✓$$

[Note that based on our drawing, y is actually increasing in the downward direction: $y=0$ is top of cone and $y=h$ is base of cone.]

The approach we've outlined here is particularly well suited to "volumes of revolution"; solids constructed by revolving a given area around an axis.

Ex: Find the volume obtained when the area bounded by the y-axis, the line $y = 9$, and the curve $y = x^2$ is rotated around the y-axis.

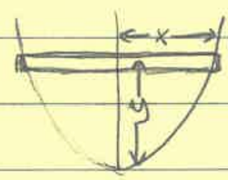


=>

Q: what should the cross-sections be? (disks)

Q: What axis should they be perpendicular to? (y-axis)

(the axis of revolution)



$dv = \pi x^2 dy$ but $y = x^2$ so

$dv = \pi y \, dy$

$$V = \int_0^9 \pi y \, dy = \pi \frac{y^2}{2} \Big|_0^9 = \frac{81}{2}\pi$$

$\boxed{\text{Volume of paraboloid is } \frac{81\pi}{2}}$

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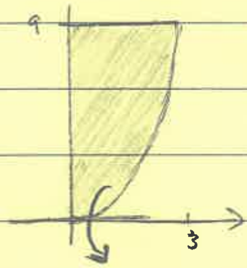
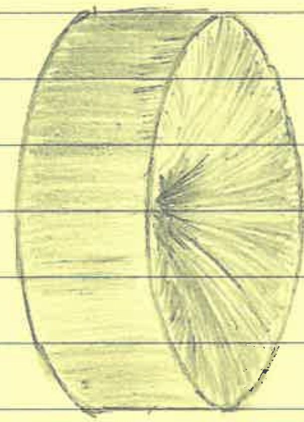
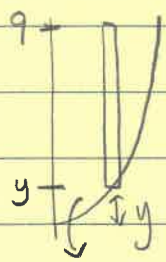
Ex. Now suppose the same area is rotated about the x-axis. Then we obtain a completely different solid:





Q: What axis should we have perpendicular to the cross-sections? (x-axis, axis of revolution)

Q: What will the cross-sections be? (washers – disks with missing centers)



dv = volume of disk – volume of hole

$dv = \pi 9^2 dx - \pi y^2 dx$

$= \pi(81 - y^2)dx$

$dv = \pi(81 - (x^2)^2)dx$

after we replace y with x^2.

limits of integration $0 \le x \le 3$

$$V = \int dv = \int_0^3 \pi(81 - x^4)dx = \pi\left(81x - \frac{x^5}{5}\right)_0^3 = \pi\left(243 - \frac{243}{5}\right)$$

$$= \frac{972}{5}\pi$$

$\boxed{\text{Volume will be } \frac{972\pi}{5}}$

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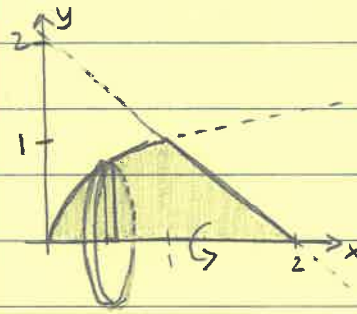
Ex What volume results when the area banded by the x-axis, the curve $y=\sqrt{x}$, and the line $y=2-x$ is rotated about a) the x-axis, b) the y-axis, c) the line $y=2$?



a) Notice that the height of the partitioning rectangles is given by $y=\sqrt{x}$ on $[0,1]$ and by $y=2-x$ on $[1,2]$.

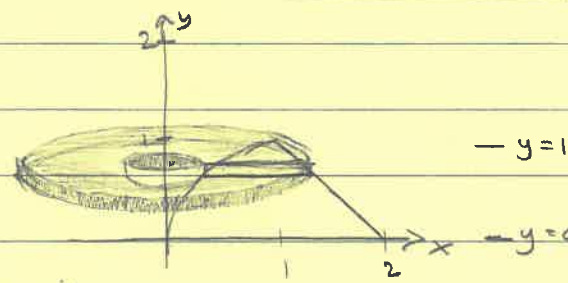
on $[0,1]$: $dv = \pi y^2 dx = \pi (\sqrt{x})^2 dx = \pi x\, dx$

on $[1,2]$: $dv = \pi y^2 dx = \pi (2-x)^2 dx = \pi(4-4x+x^2)\,dx$

$$V = \int_0^1 \pi x\, dx + \int_1^2 \pi(4-4x+x^2)\, dx = \pi \frac{x^2}{2}\Big|_0^1 + \pi\left(4x-2x^2+\frac{x^3}{3}\right)\Big|_1^2$$

$$= \frac{\pi}{2} + \pi\left[\left(8-8+\frac{8}{3}\right)-\left(4-2+\frac{1}{3}\right)\right] = \frac{\pi}{2}+\frac{\pi}{3} = \boxed{\frac{5\pi}{6} \text{ cubic units}}$$

b) When we rotate about the y-axis we have washers — the outer radius is given by $y=2-x$ or $x=2-y$, while the inner radius is given by $y=\sqrt{x}$ or $x=y^2$.

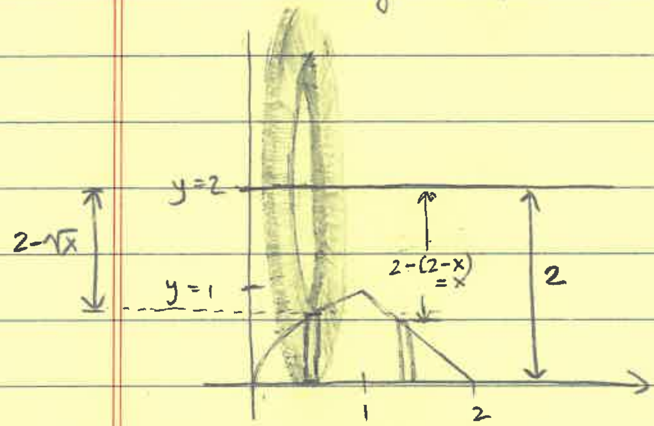


$$dv = \pi\left[x_2^2 - x_1^2\right]dy = \pi\left[(2-y)^2-(y^2)^2\right]dy$$

$$V = \int_0^1 \pi\left[4-4y+y^2-y^4\right]dy = \pi\left[4y-2y^2+\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1 = \pi\left[4-2+\frac{1}{3}-\frac{1}{5}\right]$$

$$= \frac{30+5-3}{15}\pi = \boxed{\frac{32\pi}{15} \text{ cubic units}}$$

c) Rotating about $y=2$ will require washers and treating $[0,1]$ and $[1,2]$ separately.



On $[0,1]$ we have

$$dv = \pi\left[2^2 - (2-\sqrt{x})^2\right]dx$$
$$= \pi\left[4-4+4\sqrt{x}-x\right]dx$$
$$= \pi(4\sqrt{x}-x)\,dx$$

On $[1,2]$ we have

$$dv = \pi\left[2^2-(2-(2-x))^2\right]dx$$
$$= \pi(4-x^2)\,dx$$

$$V = \int_0^1 \pi(4\sqrt{x}-x)\,dx + \int_1^2 \pi(4-x^2)\,dx$$

$$= \pi\left[4\cdot\tfrac{2}{3}x^{3/2} - \tfrac{x^2}{2}\right]_0^1 + \pi\left[4x-\tfrac{x^3}{3}\right]_1^2$$

$$= \pi\left[\tfrac{8}{3}-\tfrac{1}{2}\right] + \pi\left[(8-\tfrac{8}{3})-(4-\tfrac{1}{3})\right]$$

$$= \frac{13\pi}{6} + 4\pi - \frac{7\pi}{3}$$

$$= \frac{13+24-14}{6}\pi = \boxed{\frac{23\pi}{6} \text{ cubic units}}$$