Heavyside "Cover-up" Method for Partial Fractions

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What follows is a very terse description of an approach to dealing with partial fractions that at least appears simpler than that taught in many introductory calculus books. In each case a specific example is used to illustrate the general principle; this is not meant as a exhaustive presentation of the method.

Recall that to use partial fractions the degree of the polynomial in the numerator must be strictly less than the degree of the polynomial in the denominator. If it is not then you must first use long division to obtain a quotient polynomial and a rational function remainder; the remainder will certainly have a denominator of higher degree than the numerator and so partial fractions can be used on it.

1. Only linear factors in denominator, no repeated factors

Consider the expansion

$$\frac{g(s)}{(s-s_0)(s-s_1)} = \frac{A}{s-s_0} + \frac{B}{s-s_1}.$$

To find A, cover up the $(s-s_0)$ factor in the denominator of the left-hand-side (LHS) (i.e. pretend it's not there) and evaluate the LHS at $s = s_0$. Note that this is equivalent to multiplying through by $(s-s_0)$ before setting $s = s_0$.

To find B, cover up the $(s-s_1)$ factor in the denominator of the LHS and evaluate the LHS at $s = s_1$.

If there were more linear factors you would continue in the same fashion.

Example:

$$\frac{7s-11}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}.$$

Covering up $(s-3)$ and substituting $s = 3$ in the LHS gives $A = 2$. Covering up $(s+2)$ and substituting $s = -2$ gives $B = 5$. So

$$\frac{7s-11}{(s-3)(s+2)} = \frac{2}{s-3} + \frac{5}{s+2}.$$

2. Only linear factors in denominator, repeated factors

Consider the partial fraction expansion

$$\frac{g(s)}{(s-s_0)^2(s-s_1)} = \frac{A}{(s-s_0)^2} + \frac{B}{s-s_0} + \frac{C}{s-s_1}.$$

Start by finding A. Cover up the $(s-s_0)^2$ factor in LHS and evaluate at $s = s_0$. Once A is known, move the term with A to LHS and simplify:

$$\frac{g(s)}{(s-s_0)^2(s-s_1)} - \frac{A}{(s-s_0)^2} = \frac{B}{s-s_0} + \frac{C}{s-s_1}$$

$$\frac{g(s) - A(s-s_1)}{(s-s_0)^2(s-s_1)} = \frac{B}{s-s_0} + \frac{C}{s-s_1}$$

Now it must be the case that both numerator and denominator of the LHS have $(s-s_0)$ as a factor. Let $\bar{g}(s)$ be the function obtained by dividing $g(s) - A(s-s_1)$ by $(s-s_0)$. Then

$$\frac{\bar{g}(s)}{(s-s_0)(s-s_1)} = \frac{B}{s-s_0} + \frac{C}{s-s_1}$$

which reduces to the no repeated factors case.

Example:

$$\frac{3s^2 - 16s + 21}{(s-1)^2(s+3)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+3}$$

Covering up $(s-1)^2$ in the LHS and evaluating at $s=1$ gives $A=2$. Moving the A term to the LHS and simplifying gives

$$\frac{3s^2-16s+21}{(s-1)^2(s+3)}-\frac{2}{(s-1)^2}=\frac{3s^2-16s+21-2(s+3)}{(s-1)^2(s+3)}=\frac{3s^2-18s+15}{(s-1)^2(s+3)}$$

so

$$\frac{3s^2-18s+15}{(s-1)^2(s+3)}=\frac{B}{s-1}+\frac{C}{s+3}.$$

It must now be the case that $(s-1)$ is a factor of both the numerator and the denominator on the LHS. This can be seen since the right-hand-side (RHS) is the partial fraction expansion of a rational function with $(s-1)(s+3)$ in the denominator. In any event, dividing the numerator by $(s-1)$ leaves $3s-15$ so we have

$$\frac{3s-15}{(s-1)(s+3)}=\frac{B}{s-1}+\frac{C}{s+3}.$$

Now no linear factors in the denominator are repeated so we can quickly find $B=-3$ and $C=6$. Thus

$$\frac{3s^2-16s+21}{(s-1)^2(s+3)}=\frac{2}{(s-1)^2}-\frac{3}{s-1}+\frac{6}{s+3}.$$

3. Linear and quadratic factors in denominator

This can be combined with the above cases to handle a range of different cases.

$$\frac{g(s)}{(s^2+ps+q)(s-s_0)}=\frac{As+B}{s^2+ps+q}+\frac{C}{s-s_0}$$

Find C first by cover-up method with $s=s_0$ then move C term to the LHS and simplify:

$$\frac{g(s)-C(s^2+ps+q)}{(s^2+ps+q)(s-s_0)}=\frac{As+B}{s^2+ps+q}$$

You will find that $g(s)-C(s^2+ps+q)$ has a factor of $(s-s_0)$. After dividing this factor out you will obtain

$$\frac{\bar{g}(s)}{(s^2+ps+q)}=\frac{As+B}{s^2+ps+q}$$

and A and B can be determined by inspection. Note that if there are multiple quadratic factors this method will likely not be of much help.

Example:

$$\frac{3s}{(s^2+2)(s-1)}=\frac{As+B}{s^2+2}+\frac{C}{s-1}$$

By covering up $(s-1)$ in the LHS and substituting $s=1$ we find that $C=1$. Now move the C term to the LHS and simplify

$$\frac{3s}{(s^2+2)(s-1)}-\frac{1}{s-1}=\frac{3s-s^2-2}{(s^2+2)(s-1)}=-\frac{(s-1)(s-2)}{(s^2+2)(s-1)}=\frac{2-s}{s^2+2}$$

so

$$\frac{2-s}{s^2+2}=\frac{As+B}{s^2+2}$$

which gives $A=-1$ and $B=2$. Thus

$$\frac{3s}{(s^2+2)(s-1)}=\frac{2-s}{s^2+2}+\frac{1}{s-1}.$$