2.2 The Inverse of a Matrix

If A is $n \times n$ (square) and a matrix can be found such that

$$AX = XA = I$$

we call X the <u>inverse</u> of A (or A inverse) and denote it A^{-1}.

Q: How many inverses does a matrix have?

Assume B is any inverse of A. Then

$$BA = I$$
$$BAA^{-1} = IA^{-1}$$
$$BI = A^{-1}$$
$$B = A^{-1}$$

So A^{-1} is unique.

The inverse is very useful. Consider $A\vec{x} = \vec{b}$. If A is an $n \times n$ invertible matrix then

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

So (in theory) we can solve the linear system $A\vec{x} = \vec{b}$ by finding A^{-1} and then multiplying $\vec{b}$ on the left by A^{-1}.

Problem; while many square matrices do have inverses, the work necessary to compute them is much greater than solving $A\vec{x} = \vec{b}$ without using inverses.

That said, however, we will see that inverses play an important role and are quite useful.

For example, if A $(n \times n)$ is invertible so that $\vec{x} = A^{-1}\vec{b}$ for any $\vec{b} \in \mathbb{R}^n$. This tells us that $A\vec{x} = \vec{b}$ is consistent <u>and</u> that it has a unique solution

Thm

If A is an invertible $n \times n$ matrix, then for each $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$

Thm

A) If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$

proof: If $X = A^{-1}$ then $XA = AX = I$. But this says that X is an invertible matrix whose inverse is A. $\therefore (A^{-1})^{-1} = A$.

B) If A and B are invertible $n \times n$ matrices, so is AB. Also

$$(AB)^{-1} = B^{-1}A^{-1}$$

proof: Since A and B are invertible

$$B^{-1}A^{-1}AB = ABB^{-1}A^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = (AB)(B^{-1}A^{-1}) = I$$

So (AB) is invertible and its inverse $(AB)^{-1}$ is $B^{-1}A^{-1}$.

C) If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$

proof:

$$AA^{-1} = A^{-1}A = I$$

$$(AA^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I$$

so the inverse of A^T is $(A^{-1})^T$.

Sometimes $(A^{-1})^T$ is denoted A^{-T}

Q: Given A, how do we find A^{-1} (if it exists)?

Assuming A^{-1} exists, set $B = A^{-1}$ so that $AB = I$.

$$AB = A[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n] = [A\vec{b}_1\ A\vec{b}_2 \cdots A\vec{b}_n]$$

$$\text{but } AB = I = [\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n]$$

columns must match

So we get n linear systems

$$A\vec{b}_1 = \vec{e}_1$$
$$A\vec{b}_2 = \vec{e}_2$$
$$\vdots$$
$$A\vec{b}_n = \vec{e}_n$$

Each of these can be solved by row reducing $[A\ \vec{e}_i]$ to get $\vec{b}_i$

I.e. row reduce

$$[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n\ \vec{e}_1]$$
$$\vdots$$
$$[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n\ \vec{e}_n]$$

But we can do this all at once by row reducing

$$[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n\ \vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n]$$

or

$$[A\ I]$$

Since if A is invertible it has n-pivot columns (why?) we know A is row equivalent to I. When we row reduce $[A\ I]$ we end up with $[I\ B]$, but recall that $B = A^{-1}$.

Thm An $n \times n$ matrix A is invertible iff it is row equivalent to I_n.

<u>Proof</u> Let E be an elementary matrix, obtained by performing a single elementary row operation on I_n. Note that EA yields a matrix identical to performing the same row operation on A itself.

<u>If A is row equivalent to I_n</u> then there exist elementary matrices $E_1, E_2, \dots, E_p$ such that

$$E_p E_{p-1} \cdots E_2 E_1 A = I_n$$

In this case $A^{-1} = E_p E_{p-1} \cdots E_2 E_1$ so A^{-1} exists

<u>If A is invertible then</u> A^{-1} exists such that $A^{-1}A = I_n$

Let $E_1, E_2, \dots, E_p$ be elementary matrices such that $E_p \cdots E_1 I_n = A^{-1}$. These are the same operations that turn A into I_n since $E_p \cdots E_1 A = A^{-1}A = I_n$

Therefore A is row equivalent to I_n

Ex Find the inverse of $A=\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -3/2 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1/2 & -3/2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Check $AA^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

$A^{-1}A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 2-2 \\ -6+6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

For the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we know how to compute the inverse (and check for existence)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1}\ \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

which only exists if $ad-bc \neq 0$

Text discusses <u>elementary matrices</u>, which are obtained by performing a single row operation on I. If E is such a matrix the EA is that matrix which would result from performing the same row operation on A itself.

Discuss : 2.2: 16, 18, 22

Octave uses inv(A) to compute the inverse of A.
Sage uses A.inverse()